

A prime geodesic theorem for SL_4

Anton Deitmar & Mark Pavey

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Introduction

The prime geodesic theorem (PGT) gives a growth asymptotic for the number of closed geodesics counted by their lengths [15, 21, 27, 28, 30, 31, 33, 43]. Classically, it was known for manifolds of strictly negative sectional curvature. In [11], one finds a generalization to higher rank locally symmetric manifolds, counting only regular geodesics. In [13] the PGT is extended to non-regular geodesics, under the proviso of a geometric regularity condition on the manifold.

The current paper is dedicated to quotients of the symmetric space of $\mathrm{SL}_4(\mathbb{R})$. In this situation the regularity condition fails. As a consequence, the analytical difficulties grow exponentially. The group $\mathrm{SL}_4(\mathbb{R})$ is the “biggest” semisimple group for which the unitary dual is known. This enables us to give prime geodesic theorems with remainder term.

In [9, 12, 11, 13], the prime geodesic theorem is applied to derive class number asymptotics. The last open case for these is the case of complex quartic fields for which the PGT required is the one of $\mathrm{SL}_4(\mathbb{R})$. As there are additional algebraic obstacles, this application of the PGT is not straightforward and will be subject of a subsequent paper.

In what follows we give a summary of the arguments and techniques used in the proof of our main results. In particular we shall point out the difficulties that arise in applying the methods of [11] to our situation. Section 1 provides some necessary background and preliminary results. In Section 2 we introduce the zeta functions we shall be studying and prove some of their analytic properties. The results of Section 1 are made use of here in the definitions of the zeta functions and proofs of their analytic properties. In Section 3 we apply standard techniques from analytic number theory to the results of Section 2 in order to prove the Prime Geodesic Theorem required in our context.

Our strategy is that we define a suitable zeta function and use the trace formula with special test-functions to derive analytic continuation as in [8], so that we can use the techniques of the proof of the prime number theorem. It is in the analytic continuation that we encounter the first obstacle. We say that an element $g \in G$ is *weakly neat* if the adjoint $\mathrm{Ad}(g)$ has no non-trivial roots of unity as eigenvalues. A subgroup of G is weakly neat if every element is. The results of [8] use the assumptions that the group G has trivial centre and the group Γ is weakly neat. Note that, if G has trivial centre then Γ weakly neat implies Γ torsion free, since any non weakly neat torsion elements must be central. To generalise the results of [8] to the case where Γ is not weakly neat requires two things. Firstly, we need to modify the definition of the zeta function from that given in [8]. Secondly, the definition of the zeta function includes the first higher Euler characteristic of the spaces X_{Γ_γ} for $\gamma \in \mathcal{E}(\Gamma)$. In [8] these are only defined for Γ_γ torsion free so we need to broaden the definition.

In Section 1 we give a definition of first higher Euler characteristics which is general enough for our application. We further prove that its value is always positive in the cases we consider. This fact is needed in the proof of the Prime Geodesic Theorem. In [8] it is shown that the position of the poles and zeros of the generalised Selberg zeta function depend on the Lie algebra cohomology of the irreducible, unitary representations of $\mathrm{SL}_4(\mathbb{R})$. Using a result of Hecht and Schmid ([20]) it suffices to look at the infinitesimal characters of the irreducible, unitary representations of $\mathrm{SL}_4(\mathbb{R})$. For this purpose, in Section 1 we also describe the unitary dual of $\mathrm{SL}_4(\mathbb{R})$ using a result of Speh ([39]) and give a result about the infinitesimal characters of certain elements of this set.

In Section 2 we define the generalised Selberg zeta function and use the trace formula to deduce its analytic properties. In particular, we give a formula for its vanishing order at a given point and prove a functional equation, from which we can deduce that it is of finite order. We then define the generalised Ruelle zeta function and prove that it is a finite quotient of generalised Selberg zeta functions. In particular, in the cases we are interested in it has a zero at $s = 1$ and all other poles and zeros in the half plane $\mathrm{Re} s \leq \frac{3}{4}$, and is furthermore of finite order. We introduce the Ruelle zeta function as its logarithmic derivative is a Dirichlet series from whose properties we can prove the Prime Geodesic Theorem.

The definitions of the generalised Selberg and Ruelle zeta functions depend on the choice of a finite dimensional virtual representation of a particular closed subgroup M of $\mathrm{SL}_4(\mathbb{R})$. The trace of this representation at elements of Γ appears in the Dirichlet series arising as the logarithmic derivative of the Ruelle zeta function. We can choose such a representation $\tilde{\sigma}$ so that its trace is zero for all non-regular elements of Γ .

In Section 3 we apply the methods of [34] together with standard techniques of analytic number theory to prove a Prime Geodesic Theorem. In carrying out the application to class numbers it is necessary to show that certain subsets of the summands contribute negligibly to the asymptotic. This is done

using a “sandwiching” argument. To carry out this “sandwiching” argument we define a sequence of Dirichlet series, which are not connected to any zeta function, but whose analytic properties can also be deduced from the Selberg trace formula. We use a version of the Wiener-Ikehara theorem to prove an asymptotic result for an increasing sequence of functions derived from these Dirichlet series, which we use to “sandwich” the product over the elements we are interested in against the sum over all elements which comes from the Ruelle zeta function. Unfortunately the asymptotics we are able to derive from the Wiener-Ikehara theorem do not provide an error term like those we can deduce from the Ruelle zeta function using the methods of [34]. This is why we lose the error term in Theorem 3.2.

We finish this introduction with a few remarks about the limitations of the method used here in terms of further applications and mention a couple of other recent results in the same direction. In order to be able to make use of the trace formula for compact spaces we have had to limit our sum over class numbers by means of the choice of a finite set of primes, as described above. In [12] Deitmar and Hoffmann have been able to use a different trace formula to prove that as $x \rightarrow \infty$

$$\sum_{R(\mathcal{O}) \leq x} h(\mathcal{O}) \sim \frac{e^{3x}}{3x},$$

where the sum is over all isomorphism classes of orders in complex cubic fields.

In order to get the error term in the Prime Geodesic Theorem we have made use of the classification of the unitary dual of $\mathrm{SL}_4(\mathbb{R})$. At present the unitary dual is not known for any higher dimensional groups so a Prime Geodesic Theorem with error term is not possible using our methods. Finally we mention that the correspondence between geodesics and orders actually works by identifying primitive geodesics with fundamental units in orders. By Dirichlet’s unit theorem, an order in a number field F has a unique fundamental unit (up to inversion and multiplication by a root of unity) only if F is real quadratic, complex cubic or totally complex quartic. Hence an asymptotic of our form can be proven only in these three cases.

1 Euler Characteristics and Infinitesimal Characters

In this section we introduce some concepts and prove some results which will be needed for our consideration of the zeta functions in the next section.

1.1 The unitary duals of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_4(\mathbb{R})$

The *unitary dual* \hat{G} of a locally compact group G is the set of all equivalence classes of irreducible unitary representations of G . If π is an irreducible, unitary representation of G then we shall write, by slight abuse of notation, $\pi \in \hat{G}$. Let $G_1 = \mathrm{SL}_2(\mathbb{R})$. We define the following subgroups: let K_1 be the maximal compact subgroup $\mathrm{SO}(2)$; let $M_1 = \{\pm I_2\}$. let A_1 be the group of diagonal matrices $\mathrm{diag}(a, a^{-1})$ where $a > 0$, and let N_1 be the group of upper triangular matrices in G_1 with ones on the diagonal. Let $\mathfrak{g}_1, \mathfrak{k}_1, \mathfrak{a}_1, \mathfrak{n}_1$ be the respective complexified Lie algebras. Let $\rho_1 \in \mathfrak{a}_1^*$ be defined by $\rho(\mathrm{diag}(a, -a)) = a$, and let $P_1 = M_1 A_1 N_1$ be the parabolic subgroup of G_1 with split torus A_1 and unipotent radical N_1 . We will use the notation of [26], Chap. II. For $n \geq 2$ we have the discrete series representations \mathcal{D}_n^+ and \mathcal{D}_n^- of G_1 . We also denote the two limit of discrete series representations by \mathcal{D}_1^+ and \mathcal{D}_1^- .

Let τ^+, τ^- be the two characters of $M - 1$. For $\nu \in \mathfrak{a}_1^*$ we define $\mathcal{P}^{\pm, \nu} = \mathrm{Ind}_{P_1}^{G_1}(\tau^{\pm} \otimes \nu)$. For $x \in \mathbb{R}$ define the *principal series representations*

$$\mathcal{P}^{\pm, ix} = \mathcal{P}^{\pm, ix\rho_1}.$$

Then the representations $\mathcal{P}^{\pm, ix}$ are unitary and are all irreducible except for $\mathcal{P}^{-, 0} \cong \mathcal{D}_1^+ \oplus \mathcal{D}_1^-$. The only equivalences among the representations $\mathcal{P}^{\pm, \nu}$ are that $\mathcal{P}^{+, ix}$ and $\mathcal{P}^{-, ix}$ are equivalent to $\mathcal{P}^{+, -ix}$ and $\mathcal{P}^{-, -ix}$ respectively for all $x \in \mathbb{R}$. We also define the *complementary series representations* $\mathcal{C}^x = \mathcal{P}^{+, x\rho_1}$, for $0 < x < 1$. We have the following classification theorem:

Theorem 1.1 *The unitary dual of $\mathrm{SL}_2(\mathbb{R})$ consists of*

- a) *the trivial representation;*
- b) *the principal series representations $\mathcal{P}^{+, ix}$ for $x \in \mathbb{R}$ and $\mathcal{P}^{-, ix}$ for $x \in \mathbb{R} \setminus \{0\}$;*
- c) *the complementary series \mathcal{C}^x for $0 < x < 1$;*
- d) *the discrete series \mathcal{D}_n^+ and \mathcal{D}_n^- for $n \geq 2$ and limits of discrete series \mathcal{D}_1^+ and \mathcal{D}_1^- .*

The only equivalences among these representations are $\mathcal{P}^{\pm, ix} \cong \mathcal{P}^{\pm, -ix}$ for all $x \in \mathbb{R}$.

Proof: [26], Theorem 16.3. □

Now let $G = \mathrm{SL}_4(\mathbb{R})$ and $K = \mathrm{SO}(4)$, so K is a maximal compact subgroup of G . Let $P' = M'A'N'$ be a parabolic subgroup of G with split component A' and unipotent radical N' . Let \mathfrak{g} and \mathfrak{a}' be the complexified Lie algebras of G and A' and let \mathfrak{a}'^* be the complex dual of \mathfrak{a}' . Let ρ' be the half sum of the positive roots of the system $(\mathfrak{g}', \mathfrak{a}')$. Then we can define induced representations in an entirely analogous way to that used for $G_1 = \mathrm{SL}_2(\mathbb{R})$ above. For an irreducible, unitary representation τ of M' and for $\nu \in \mathfrak{a}'^*$ we write the corresponding induced representation of G as $\mathrm{Ind}_{P'}^G(\tau \otimes \nu)$. If $\nu = ix\rho'$ for some $x \in \mathbb{R}$ then the induced representation $\mathrm{Ind}_{P'}^G(\tau \otimes \nu)$ is unitary with respect to the inner product $\langle f, g \rangle = \int_K \langle f(k), g(k) \rangle_{\tau} dk$. In this case we call $\mathrm{Ind}_{P'}^G(\tau \otimes \nu)$ a principal series representation. For other choices is $\nu \in \mathfrak{a}'^*$ it may be possible to make $\mathrm{Ind}_{P'}^G(\tau \otimes \nu)$ unitary with respect to a different inner product, in which case we call $\mathrm{Ind}_{P'}^G(\tau \otimes \nu)$ a complementary series representation. There are also certain irreducible, unitary subrepresentations of induced representations (see [39], p121) which are called *limit of complementary series representations*. These limit of complementary series representations can often be realised as induced representations from a parabolic subgroup $P'' \supset P'$.

We define the following subgroups of G . Let

$$\begin{aligned} M &= \mathrm{S} \left(\begin{array}{c} \mathrm{SL}_2^{\pm}(\mathbb{R}) \\ \mathrm{SL}_2^{\pm}(\mathbb{R}) \end{array} \right) \\ &\cong \left\{ (x, y) \in \mathrm{Mat}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{R}) \mid \begin{array}{l} \det(x), \det(y) = \pm 1 \\ \det(x)\det(y) = 1 \end{array} \right\}. \end{aligned}$$

Let A be the group of all diagonal matrices $\mathrm{diag}(a, a, a^{-1}, a^{-1})$, where $a > 0$. Finally, let $N = \left(\begin{array}{cc} I_2 & \mathrm{Mat}_2(\mathbb{R}) \\ 0 & I_2 \end{array} \right)$. Let P be the parabolic subgroup of G with Langlands decomposition $P = MAN$. For

$m_1, m_2 \in \mathbb{N}$, we denote by $\bar{\pi}_{m_1, m_2}$ the representation of M induced from the representation $\mathcal{D}_{m_1}^+ \otimes \mathcal{D}_{m_2}^+$ of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. For $m \in \mathbb{N}$, let $\bar{\pi}_m = \bar{\pi}_{m, m}$ and let $I_m = \mathrm{Ind}_P^G(\bar{\pi}_m \otimes \frac{1}{2}\rho_P)$. The representations I_m each have a unique irreducible quotient known as the Langlands quotient (see [26], Theorem 7.24). We denote the Langlands quotient of I_m by π_m .

We have the following classification theorem:

Theorem 1.2 *The unitary dual of $\mathrm{SL}_4(\mathbb{R})$ consists of*

- a) *the trivial representation;*
- b) *principal series representations;*
- c) *complementary series representations $\mathrm{Ind}_P^G(\bar{\pi}_m \otimes t\rho_P)$, for $m \in \mathbb{N}$ and $0 < t < \frac{1}{2}$;*
- d) *complementary series representations induced from parabolics other than $P = MAN$;*
- e) *limit of complementary series representations, which are irreducible unitary subrepresentations of I_m , for $m \in \mathbb{N}$;*
- f) *the family of representations π_m , indexed by $m \in \mathbb{N}$.*

Proof: This follows from [39], Theorem 5.1, where the unitary dual of $\mathrm{SL}_4^\pm(\mathbb{R}) = \{X \in \mathrm{Mat}_4(\mathbb{R}) : \det X = \pm 1\}$ is given. \square

1.2 Euler characteristics

Let G be a real reductive group and suppose that there is a finite subgroup E of the centre of G and a reductive and Zariski-connected linear group \mathcal{G} such that G/E is isomorphic to a subgroup of $\mathcal{G}(\mathbb{R})$ of finite index. Note these conditions are satisfied whenever G is a Levi component of a connected semisimple group with finite centre. Let K be a maximal compact subgroup of G and let Γ be a discrete, cocompact subgroup of G . Let X_Γ be the locally symmetric space $\Gamma \backslash G/K$. If Γ is torsion-free we define the *first higher Euler characteristic* of Γ to be

$$\chi_1(X_\Gamma) = \chi_1(\Gamma) = \sum_{j=0}^{\dim X_\Gamma} (-1)^{j+1} j h^j(X_\Gamma),$$

where $h^j(X_\Gamma)$ is the j th Betti number of X_Γ . We want to generalise this definition to cases where the group Γ is not necessarily torsion free. We prove below a proposition which allows us broaden the definition to all cases required by our applications.

Let θ be the Cartan involution fixing K pointwise. There exists a θ -stable Cartan subgroup $H = AB$ of G , where A is a connected split torus and $B \subset K$ is a Cartan of K . We assume that A is central in G . Let C denote the centre of G . Then $C \subset H$ and we write B_C, Γ_C for $B \cap C$ and $\Gamma \cap C$ respectively. Let G^1 be the derived group of G and let $\Gamma^1 = G^1 \cap \Gamma_C$. We note in particular that, since $G = G^1 C$, we have $\Gamma \subset \Gamma^1 C \subset \Gamma^1 AB$. Let $\Gamma_A = A \cap \Gamma_C B_C$ be the projection of Γ_C to A . Then Γ_A is discrete and cocompact in A (see [42], Lemma 3.3).

Let $\mathfrak{g}_\mathbb{R}$ be the real Lie algebra $\mathrm{Lie}(G)$ with polar decomposition $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} \oplus \mathfrak{p}_\mathbb{R}$. Let b be a fixed non-degenerate invariant bilinear form on the Lie algebra $\mathfrak{g}_\mathbb{R}$ which is negative definite on $\mathfrak{k}_\mathbb{R}$ and positive definite on $\mathfrak{p}_\mathbb{R}$. Then the form $-b(X, \theta(Y))$ is positive definite and thus defines a left invariant metric on G . Unless otherwise stated all Haar measures will be normalised according to the Harish-Chandra normalisation given in [19], §7. Note that this normalisation depends on the choice of the bilinear form b on \mathfrak{g} . On the space G/H we have a pseudo-Riemannian structure given by the form b . The Gauss-Bonnet construction generalises to pseudo-Riemannian structures to give an Euler-Poincaré measure η on G/H . Define a (signed) Haar measure on G by

$$\mu_{EP} = \eta \otimes (\text{Haar measure on } H).$$

The *Weyl group* $W = W(G, H)$ is defined to be the quotient of the normaliser of H in G by the centraliser. It is a finite group generated by elements of order two. We define the *generic Euler characteristic* by

$$\chi_{\mathrm{gen}}(\Gamma) = \chi_{\mathrm{gen}}(X_\Gamma) = \frac{\mu_{EP}(\Gamma \backslash G)}{|W|}.$$

Proposition 1.3 *Assume Γ is torsion free, A is central in G of dimension one and $\Gamma' \subset \Gamma$ is of finite index in Γ . Then A/Γ_A acts freely on X_Γ and $\chi_{\text{gen}}(X_\Gamma) = \chi_1(\Gamma)\text{vol}(A/\Gamma_A)$. It follows that*

$$\chi_1(\Gamma) = \chi_1(\Gamma') \frac{[\Gamma_A : \Gamma'_A]}{[\Gamma : \Gamma']}.$$

Proof: The group $A_\Gamma = A/\Gamma_A$ acts on $\Gamma \backslash G/B$ by multiplication from the right. We claim that this action is free, i.e., that it defines a fibre bundle

$$A_\Gamma \rightarrow \Gamma \backslash G/B \rightarrow \Gamma \backslash G/H.$$

To see this let $\Gamma x a B = \Gamma x B$ for some $a \in A$ and $x \in G$. Then $a = x^{-1} \gamma x b$ for some $\gamma \in \Gamma$ and $b \in B$. Since $\Gamma \subset \Gamma^1 C \subset \Gamma^1 A B$ we can write γ as $\gamma^1 a_\gamma b_\gamma$, with $\gamma^1 \in \Gamma^1$ and $a_\gamma \in A$ and $b_\gamma \in B_C = B \cap C$. It follows that $a_\gamma \in \Gamma_A$. Since A is central in G , we can write $\gamma^1 = a a_\gamma^{-1} x b^{-1} x^{-1} b_\gamma^{-1}$. Since $\gamma^1 \in G^1$ and $a a_\gamma^{-1} \in A \subset C$, we must have $a a_\gamma^{-1} = 1$, so $a = a_\gamma \in \Gamma_A$, which implies the claim. In the same way we see that we get a fibre bundle

$$A_\Gamma \rightarrow \Gamma \backslash G/K \rightarrow A\Gamma \backslash G/K. \quad (1)$$

Let χ be the usual Euler characteristic. From [24] we take the equation $\chi(K/B) = |W|$. Using multiplicativity of Euler numbers in the fibre bundle $K/B \rightarrow A\Gamma \backslash G/B \rightarrow A\Gamma \backslash G/K$ we get

$$\begin{aligned} \chi_{\text{gen}}(X_\Gamma) &= \text{vol}(A/\Gamma_A) \frac{\eta(\Gamma \backslash G/H)}{|W|} \\ &= \text{vol}(A/\Gamma_A) \frac{\chi(\Gamma \backslash G/H)}{\chi(K/B)} \\ &= \text{vol}(A/\Gamma_A) \frac{\chi(A\Gamma \backslash G/B)}{\chi(K/B)} \\ &= \text{vol}(A/\Gamma_A) \chi(A\Gamma \backslash G/K) \\ &= \text{vol}(A/\Gamma_A) \chi(A \backslash X_\Gamma). \end{aligned}$$

It remains to show that $\chi(A \backslash X_\Gamma) = \chi_1(\Gamma)$.

Let $\mathfrak{a}_\mathbb{R}$ and $\mathfrak{g}_\mathbb{R}^1$ be the Lie algebras of A and G^1 respectively. Then we can write

$$\mathfrak{g}_\mathbb{R} = \mathfrak{a}_\mathbb{R} \oplus \mathfrak{g}_\mathbb{R}^1 \oplus \mathfrak{z}_\mathbb{R}, \quad (2)$$

where $\mathfrak{z}_\mathbb{R}$ is central in $\mathfrak{g}_\mathbb{R}$. Let X be the bi-invariant vector field on $\Gamma \backslash G/K$ generating the A_Γ action. Then we can consider X as an element of $\mathfrak{a}_\mathbb{R}$ under the decomposition (2). Considering the dual of (2), we can identify $\mathfrak{a}_\mathbb{R}^*$ with a subspace of $\mathfrak{g}_\mathbb{R}^*$. Since \mathfrak{a} is central in \mathfrak{g} , a non-zero element of $\mathfrak{a}_\mathbb{R}^*$ gives us an A_Γ -invariant, closed 1-form ω on $\Gamma \backslash G/K$ such that for every $\Gamma g K \in \Gamma \backslash G/K$ we have $\omega(\Gamma g K)(X) \neq 0$. Since $A_\Gamma \cong \mathbb{R}/\mathbb{Z}$ is connected and compact, the cohomology of the de Rham complex $\Gamma \backslash G/K$ coincides with the cohomology of the subcomplex of A_Γ -invariants $\Omega(\Gamma \backslash G/K)^{A_\Gamma}$. Using local triviality of the bundle one sees that

$$\Omega(\Gamma \backslash G/K)^{A_\Gamma} = \pi^* \Omega(A\Gamma \backslash G/K) \oplus \pi^* \Omega(A\Gamma \backslash G/K) \wedge \omega,$$

where π^* denotes the projection map. Set $C_0 = \pi^* \Omega(A\Gamma \backslash G/K)$ and $C_1 = C_0 \oplus C_0 \wedge \omega = \Omega(\Gamma \backslash G/K)^{A_\Gamma}$. Then

$$H^p(C_1) = H^p(C_0) \oplus H^{p-1}(C_0)$$

and so

$$\begin{aligned} \chi_1(C_1) &= \sum_{p \geq 0} (-1)^{p+1} p \dim H^p(C_1) \\ &= \sum_{p \geq 0} (-1)^{p+1} p (\dim H^p(C_0) + \dim H^{p-1}(C_0)) \\ &= \sum_{p \geq 0} (-1)^{p+1} (p - (p+1)) \dim H^p(C_0) \\ &= \sum_{p \geq 0} (-1)^p \dim H^p(C_0) = \chi(C_0). \end{aligned}$$

This gives the required result. \square

Let Γ be a discrete, cocompact subgroup of G . Then Γ has a torsion-free subgroup Γ' of finite index. Suppose that the torus A is central in G and of dimension one. Define Γ_A and Γ'_A as above.

We define the first higher Euler characteristic of Γ as

$$\chi_1(X_\Gamma) = \chi_1(\Gamma) = \chi_1(\Gamma') \frac{[\Gamma_A : \Gamma'_A]}{[\Gamma : \Gamma']}. \quad (3)$$

Proposition 1.3 shows that this is well-defined. We note that in the case that Γ itself is torsion free, this definition of first higher Euler characteristic agrees with the one given above.

1.3 Euler-Poincaré functions

For the next definition we assume G to be a semisimple real reductive group of inner type and we fix a maximal compact subgroup K . We further assume that G contains a compact Cartan subgroup. Let $\mathfrak{g}_\mathbb{R}$ be the real Lie algebra $\text{Lie}(G)$ with polar decomposition $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} \oplus \mathfrak{p}_\mathbb{R}$, and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for its complexification. Recall that we are using Harish-Chandra's Haar measure normalisation as given in [19], §7 and this normalisation depends on the choice of an invariant bilinear form b on \mathfrak{g} , which for our purposes in this section we shall leave arbitrary.

A *virtual representation* σ of a group is a formal difference of two representations $\sigma = \sigma^+ - \sigma^-$, which is called finite dimensional if both σ^+ and σ^- are. Two virtual representations $\sigma = \sigma^+ - \sigma^-$ and $\tau = \tau^+ - \tau^-$ of a group are said to be isomorphic if there is an isomorphism $\sigma^+ \oplus \tau^- \cong \tau^+ \oplus \sigma^-$. The trace and determinant of a virtual representation $\sigma = \sigma^+ - \sigma^-$ are defined by $\text{tr } \sigma = \text{tr } \sigma^+ - \text{tr } \sigma^-$ and $\det \sigma = \det \sigma^+ / \det \sigma^-$. The dimension of σ is defined as $\dim \sigma = \dim \sigma^+ - \dim \sigma^-$.

If V is a representation space with \mathbb{Z} -grading then we shall consider it naturally as a virtual representation space by $V^+ = V_{\text{even}}$ and $V^- = V_{\text{odd}}$. In particular, if V is a subspace of \mathfrak{g} we shall always consider the exterior product $\bigwedge^* V$ as a virtual representation $\bigwedge^* V = \bigwedge^{\text{even}} V - \bigwedge^{\text{odd}} V$ with respect to the adjoint representation. We consider symmetric powers and cohomology spaces similarly.

For a smooth function f on G of compact support and an irreducible unitary representation $(\pi, V_\pi) \in \hat{G}$ define the operator

$$\pi(f) = \int_G \pi(g) f(g) dg$$

on V_π . Since f is smooth and has compact support, $\pi(f)$ is of trace class.

Let (σ, V_σ) be a finite dimensional virtual representation of G . An *Euler-Poincaré function* f_σ for σ is a compactly supported, smooth function on G such that $f_\sigma(kxk^{-1}) = f_\sigma(x)$ for all $k \in K$ and for every irreducible unitary representation (π, V_π) of G the following identity holds:

$$\text{tr } \pi(f_\sigma) = \sum_{p=0}^{\dim(\mathfrak{p})} (-1)^p \dim \left(V_\pi \otimes \bigwedge^p \mathfrak{p} \otimes V_\sigma \right)^K, \quad (4)$$

where the superscript K denotes the subspace of K invariants. By [29] such functions exist. We note that the value of such functions depends on the choice of Haar measure. We shall assume all Euler-Poincaré functions are given with respect to the Harish-Chandra normalisation. In the following lemmas we prove some of their properties.

Lemma 1.4 *Let G denote a semisimple real reductive group of inner type, with connected component G^0 , maximal compact subgroup K and compact Cartan subgroup $T \subset K$. Let $G^+ = TG^0$. Further let σ be a finite dimensional representation of G , $\sigma^+ = \sigma|_{G^+}$ and f_σ an Euler-Poincaré function for σ on G . Then $f_\sigma|_{G^+}$ is an Euler-Poincaré function for σ^+ on G^+ .*

Proof: This is Lemma 1.5 of [8]. \square

Lemma 1.5 *Let H, H_1, H_2 be real reductive groups of inner type such that $H = H_1 \times H_2$. Let σ be an irreducible representation of H . There exist irreducible representations σ_1, σ_2 of H_1, H_2 respectively such that $\sigma \cong \sigma_1 \otimes \sigma_2$ and let f_{σ_i} be an Euler-Poincaré function for σ_i on H_i .*

Then $f_\sigma(h_1, h_2) = f_{\sigma_1}(h_1) f_{\sigma_2}(h_2)$ is an Euler-Poincaré function for σ on H .

Proof: This is a straightforward computation. \square

Lemma 1.6 *Let G denote a semisimple real reductive group of inner type, with maximal compact subgroup K and compact Cartan subgroup $T \subset K$. Let $g \in G$ be central. Let σ be a finite dimensional representation of G and let f_σ, h_σ be Euler-Poincaré functions for σ on G . Then $f_\sigma(g) = h_\sigma(g)$.*

Proof: Since g is central, the orbital integral can be written as

$$\mathcal{O}_g(f_\sigma) = \int_{G_g \backslash G} f_\sigma(x^{-1}gx) dx = f_\sigma(g),$$

where G_g denotes the centraliser of g in G (which in this case, since g is central, is the whole of G). Similarly $\mathcal{O}_g(h_\sigma) = h_\sigma(g)$. By [8], Proposition 1.4, the value of the orbital integral $\mathcal{O}_g(f_\sigma)$ of an Euler-Poincaré function for σ on G depends only on g , not on the particular Euler-Poincaré function chosen. Hence, $f_\sigma(g) = \mathcal{O}_g(f_\sigma) = \mathcal{O}_g(h_\sigma) = h_\sigma(g)$, as claimed. \square

Lemma 1.7 *Let g_1 be an Euler-Poincaré function for the trivial representation on $\mathrm{SL}_2(\mathbb{R})$. Then $g_1(1) = g_1(-1) \in \mathbb{R}$.*

Proof: For $v \in \mathbb{R}$ let $\mathcal{P}^{+,iv}$ and $\mathcal{P}^{-,iv}$ be principal series representations on $\mathrm{SL}_2(\mathbb{R})$. For $n \in \mathbb{N}, n \geq 2$ let \mathcal{D}_n^+ and \mathcal{D}_n^- be discrete series representations on $\mathrm{SL}_2(\mathbb{R})$ and write \mathcal{D}_n^\pm for $\mathcal{D}_n^+ \oplus \mathcal{D}_n^-$. By [26], Theorem 11.6 there is a constant $M > 0$ such that for any compactly supported, smooth function f on $\mathrm{SL}_2(\mathbb{R})$ we have

$$\begin{aligned} f(1) &= M \sum_{n=2}^{\infty} (n-1) \mathrm{tr} \mathcal{D}_n^\pm(f) \\ &\quad + \frac{M}{4} \int_{-\infty}^{\infty} \mathrm{tr} \mathcal{P}^{+,iv}(f) v \tanh\left(\frac{\pi v}{2}\right) + \mathrm{tr} \mathcal{P}^{-,iv}(f) v \coth\left(\frac{\pi v}{2}\right) dv. \end{aligned}$$

Lemma 1.3 of [8] tells us that $\mathrm{tr} \mathcal{P}^{\pm,iv}(g_1) = 0$ for all $v \in \mathbb{R}$, so we have

$$g_1(1) = M \sum_{n=2}^{\infty} (n-1) \mathrm{tr} \mathcal{D}_n^\pm(g_1). \quad (5)$$

By the definition of an Euler-Poincaré function, for all $n \geq 2$

$$\mathrm{tr} \mathcal{D}_n^\pm(g_1) = \sum_{p=0}^2 (-1)^p \dim \left(\mathcal{D}_n^\pm \otimes \bigwedge^p \mathfrak{p} \right)^{\mathrm{SO}(2)},$$

where \mathfrak{p} is the complex Lie algebra

$$\mathfrak{p} = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

Hence $g_1(1) \in \mathbb{R}$.

We want to know for which values of n the trace $\mathrm{tr} \mathcal{D}_n^\pm(g_1)$ is non-zero. For this we need to know the $\mathrm{SO}(2)$ -types of \mathcal{D}_n^\pm and \mathfrak{p} .

For $l \in \mathbb{Z}$, define the one dimensional representation ε_l of $\mathrm{SO}(2)$ by

$$\varepsilon_l R(\theta) = e^{il\theta}, \quad \text{where} \quad R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in \mathrm{SO}(2).$$

Note that ε_0 is the trivial representation, which we shall denote by *triv*.

The unitary dual of $\mathrm{SO}(2)$ is the set $\{\varepsilon_l : l \in \mathbb{Z}\}$.

We have the following isomorphisms of $\mathrm{SO}(2)$ -modules:

$$\bigwedge^0 \mathfrak{p} = \text{triv}, \quad \bigwedge^1 \mathfrak{p} = \varepsilon_2 \oplus \varepsilon_{-2}, \quad \bigwedge^2 \mathfrak{p} = \text{triv}.$$

Lemma 1.8 For $n \in \mathbb{N}$ we have an isomorphism of $\mathrm{SO}(2)$ -modules:

$$\mathcal{D}_n^\pm \cong \bigoplus_{\substack{|j| \geq n \\ j \equiv n \pmod{2}}} \varepsilon_j.$$

Proof: Let τ_n be the unique n -dimensional representation of $\mathrm{SL}_2(\mathbb{R})$ (see [26], Chapter II §1). It follows from the definition of τ_n that the following isomorphism of $\mathrm{SO}(2)$ -modules holds:

$$\tau_n \cong \bigoplus_{\substack{|j| \leq n-1 \\ j \equiv n-1 \pmod{2}}} \varepsilon_j.$$

Let $P_1 = M_1 A_1 N_1$ be the minimal parabolic of $\mathrm{SL}_2(\mathbb{R})$. Then the unitary dual of $M_1 \cong \{1, -1\}$ consists of two one dimensional representations, which we denote by $1 = \text{triv}$ and -1 . We denote by ρ_1 the character of A_1

$$\rho_1 \left(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right) = a,$$

and write $\pi_{\pm 1, n-1}$ for $\mathrm{Ind}_{\bar{P}} \pm 1 \otimes (n-1) \rho_{\bar{P}}$. Then we have the following exact sequences of $\mathrm{SL}_2(\mathbb{R})$ -modules (see [26], Chapter II §5):

$$0 \rightarrow \mathcal{D}_n^\pm \rightarrow \pi_{1, n-1} \rightarrow \tau_{n-1} \rightarrow 0, \quad n \in \mathbb{N}, n \text{ even}$$

and

$$0 \rightarrow \mathcal{D}_n^\pm \rightarrow \pi_{-1, n-1} \rightarrow \tau_{n-1} \rightarrow 0, \quad n \in \mathbb{N}, n \text{ odd}, n \neq 1$$

and the isomorphism of $\mathrm{SL}_2(\mathbb{R})$ -modules

$$\mathcal{D}_1^\pm \cong \pi_{-1, 0}.$$

From the so-called *compact picture* of induced representations given in [26], Chapter VII §1, it is fairly straightforward to compute the following isomorphisms of $\mathrm{SO}(2)$ -modules, where the sums are over all integers with the same parity:

$$\pi_{1, n-1} \cong \bigoplus_{j \text{ even}} \varepsilon_j, \quad \pi_{-1, n-1} \cong \bigoplus_{j \text{ odd}} \varepsilon_j.$$

The lemma then follows easily by combining the various elements of the proof. \square

From the previous two lemmas we can see that $\mathrm{tr} \mathcal{D}_n^\pm(g_1)$ is non-zero only if $n = 2$ and in that case $\mathrm{tr} \mathcal{D}_2^\pm(g_1) = -2$, so $g_1(1) = -2M$.

It remains to show that $g_1(-1) = g_1(1)$. Let R_z be the right multiplication operator of $\mathrm{SL}_2(\mathbb{R})$ on the space $C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$ of smooth, compactly supported functions on $\mathrm{SL}_2(\mathbb{R})$. That is, $R_z g(x) = g(xz)$ for all $x, z \in \mathrm{SL}_2(\mathbb{R})$ and $g \in C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$. Let π be an irreducible, unitary representation of $\mathrm{SL}_2(\mathbb{R})$. The matrix $-1 = -I_2$ is central in $\mathrm{SL}_2(\mathbb{R})$ so $\pi(-1)$ commutes with $\pi(x)$ for all $x \in \mathrm{SL}_2(\mathbb{R})$ and hence, by Schur's Lemma ([26], Proposition 1.5) is scalar. This means that for any irreducible, unitary representation π on $\mathrm{SL}_2(\mathbb{R})$ we have $\mathrm{tr} \pi(R_{-1} g_1) = \pi(-1) \mathrm{tr} \pi(g_1)$. Thus we get

$$g_1(-1) = R_{-1} g_1(1) = M \sum_{n=2}^{\infty} (n-1) \mathcal{D}_n^\pm(-1) \mathrm{tr} \mathcal{D}_n^\pm(g_1) = \mathcal{D}_2^\pm(-1) g_1(1),$$

where $\mathcal{D}_2^\pm(-1)$ is a scalar, which we now compute. For $g \in \mathcal{D}_2^+$ and $z \in \mathbb{C}$ we have the action

$$\mathcal{D}_2^+ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) g(z) = (-bz + d)^{-2} g \left(\frac{az - c}{-bz + d} \right).$$

Hence

$$\mathcal{D}_2^+ \left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \right) g(z) = (-1)^{-2} g \left(\frac{-z}{-1} \right) = g(z).$$

Similarly we get

$$\mathcal{D}_2^- \left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \right) g(z) = g(z)$$

so $\mathcal{D}_2^\pm(-1) = 1$ and the lemma is proved. \square

1.4 Euler characteristics in the case of $\mathrm{SL}_4(\mathbb{R})$

We now consider the particular case $G = \mathrm{SL}_4(\mathbb{R})$. Let $K = \mathrm{SO}(4)$, a maximal compact subgroup of G and let Γ be a discrete, cocompact subgroup of G . Let A be the rank one torus

$$A = \left\{ \begin{pmatrix} a & & & \\ & a & & \\ & & a^{-1} & \\ & & & a^{-1} \end{pmatrix} : a > 0 \right\},$$

and let B be the compact group

$$B = \begin{pmatrix} \mathrm{SO}(2) & \\ & \mathrm{SO}(2) \end{pmatrix}.$$

B is a compact Cartan subgroup of

$$\begin{aligned} M &= \mathrm{S} \begin{pmatrix} \mathrm{SL}_2^\pm(\mathbb{R}) & \\ & \mathrm{SL}_2^\pm(\mathbb{R}) \end{pmatrix} \\ &\cong \left\{ (x, y) \in \mathrm{Mat}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{R}) \mid \begin{array}{l} \det(x), \det(y) = \pm 1 \\ \det(x)\det(y) = 1 \end{array} \right\}. \end{aligned}$$

Let

$$N = \begin{pmatrix} I_2 & \mathrm{Mat}_2(\mathbb{R}) \\ 0 & I_2 \end{pmatrix}$$

and let $P = MAN$ be a parabolic subgroup of G with Levi component MA and unipotent radical N . Let

$$A^- = \left\{ \begin{pmatrix} a & & & \\ & a & & \\ & & a^{-1} & \\ & & & a^{-1} \end{pmatrix} : 0 < a < 1 \right\},$$

be the negative Weyl chamber in A with respect to the root system given by the choice of parabolic. Let $\mathcal{E}_P(\Gamma)$ be the set of Γ -conjugacy classes of elements $\gamma \in \Gamma$ which are conjugate in G to an element of A^-B . We shall use this notation for the rest of this section.

We say $g \in G$ is *regular* if its centraliser is a torus and *non-regular* (or *singular*) otherwise. Clearly, for $\gamma \in \Gamma$ regularity is a property of the Γ -conjugacy class $[\gamma]$. Let $[\gamma] \in \mathcal{E}_P(\Gamma)$ and write G_γ for the centraliser of γ in G . The element γ is conjugate in G to an element $a_\gamma b_\gamma \in A^-B$ and we define the length l_γ of γ to be $l_\gamma = b(\log a_\gamma, \log a_\gamma)^{1/2}$. It follows that if γ is regular then $G_\gamma \cong AB$.

Let K_γ be a maximal compact subgroup of G_γ , containing B . Then the group B is a Cartan subgroup of K_γ , the product AB is a θ -stable Cartan subgroup of G_γ and A is central in G_γ . If we let $\Gamma_\gamma = \Gamma \cap G_\gamma$ denote the centraliser of γ in Γ then Γ_γ is discrete and cocompact in G_γ . Let Γ' be a torsion free subgroup of finite index in Γ and let $\Gamma'_\gamma = \Gamma' \cap G_\gamma$. Then Γ'_γ is a torsion free subgroup of finite index in Γ_γ . We define $\Gamma_{\gamma,A}$, $\Gamma'_{\gamma,A}$ as in Section 1.2. The first higher Euler characteristic $\chi_1(\Gamma_\gamma)$ of Γ_γ is then defined as in (3) as:

$$\chi_1(\Gamma_\gamma) = \frac{[\Gamma_{\gamma,A} : \Gamma'_{\gamma,A}]}{[\Gamma_\gamma : \Gamma'_\gamma]} \chi_1(\Gamma'_\gamma). \quad (6)$$

Lemma 1.9 *For $\gamma \in \mathcal{E}_P(\Gamma)$ we have that*

$$\chi_1(\Gamma_\gamma) = \frac{C_\gamma \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma)}{l_{\gamma_0}},$$

where C_γ is an explicit constant depending only on γ , which is equal to one when γ is regular, and γ_0 is the primitive geodesic underlying γ .

Proof: Since Γ'_γ is torsion free we may take from the second proposition in section 2.4 of [7] the equation

$$\chi_1(\Gamma'_\gamma) = \frac{C_\gamma \mathrm{vol}(\Gamma'_\gamma \backslash G_\gamma)}{\lambda'_\gamma},$$

where λ'_γ denotes the volume of the maximal compact flat in $\Gamma' \backslash G/K$ containing γ , and C_γ is an explicit constant depending only on γ . We note that in [7] there is an extra factor in the equation which does not

show up here. The reason is that in [7] a differential form, not a measure, was constructed. The value of the constant C_γ is given in [7] in terms of the root system of G_γ with respect to a Cartan subgroup. In the case that γ is regular, as noted above we have that $G_\gamma = AB$ and hence G_γ is a Cartan subgroup of itself. It is then easy to see that value of C_γ in this case is one.

We denote by λ_γ the volume of the maximal compact flat in $\Gamma \backslash G/K$ containing γ . The values of λ_γ and λ'_γ are given by the volumes of the images of $\Gamma_{\gamma,A} \backslash A$ and $\Gamma'_{\gamma,A} \backslash A$ respectively under their embeddings into $\Gamma \backslash G/K$ and $\Gamma' \backslash G/K$ respectively. Since A is one-dimensional, $\lambda_\gamma = l_{\gamma_0}$. Furthermore, from the definition of the groups $\Gamma_{\gamma,A}$ and $\Gamma'_{\gamma,A}$ it follows that

$$[\Gamma_{\gamma,A} : \Gamma'_{\gamma,A}] = \lambda'_\gamma / \lambda_\gamma = \lambda'_\gamma / l_{\gamma_0}.$$

Since

$$\text{vol}(\Gamma'_\gamma \backslash G_\gamma) = [\Gamma_\gamma : \Gamma'_\gamma] \text{vol}(\Gamma_\gamma \backslash G_\gamma),$$

the lemma follows from (6). \square

Proposition 1.10 *For all $\gamma \in \mathcal{E}_P(\Gamma)$ the Euler characteristic $\chi_1(\Gamma_\gamma)$ is positive. If γ is regular then*

$$\chi_1(\Gamma_\gamma) = \frac{[\Gamma_{\gamma,A} : \Gamma'_{\gamma,A}]}{[\Gamma_\gamma : \Gamma'_\gamma]}.$$

Proof: If γ is regular then $G_\gamma \cong AB$, $K_\gamma \cong B$ and Γ'_γ is a complete lattice in G_γ so $X_{\Gamma'_\gamma} = \Gamma'_\gamma \backslash G_\gamma / K_\gamma \cong S^1$, the unit circle in \mathbb{C} . The Betti numbers $h^j(S^1)$ are equal to zero for $j \neq 0, 1$ and one for $j = 0, 1$, hence $\chi_1(\Gamma'_\gamma) = 1$. The claimed value for $\chi_1(\Gamma_\gamma)$ then follows immediately from (6).

Suppose γ is not regular. Since $\gamma \in \mathcal{E}_P(\Gamma)$, we have that γ is conjugate in G to $a_\gamma b_\gamma \in A^- B \subset AM$. Let (σ, V_σ) be a finite dimensional representation of M and let K_M be the maximal compact subgroup $S(O(2) \times O(2))$ of M , which contains the compact Cartan B of M . We saw above that there exist Euler-Poincaré functions of σ on M . Let f_σ be one such. We denote by M_γ the centraliser of b_γ in M and by $\mathcal{O}_{b_\gamma}^M(f_\sigma)$ the orbital integral

$$\int_{M/M_\gamma} f_\sigma(x b_\gamma x^{-1}) dx.$$

From Proposition 1.4 of [8] we get the equation

$$\mathcal{O}_{b_\gamma}^M(f_\sigma) = C_\gamma \text{tr } \sigma(b_\gamma),$$

where C_γ is the constant from Lemma 1.9. Together with Lemma 1.9 this gives us

$$\chi_1(\Gamma_\gamma) = \frac{\mathcal{O}_{b_\gamma}^M(f_\sigma) \text{vol}(\Gamma_\gamma \backslash G_\gamma)}{l_{\gamma_0} \text{tr } \sigma(b_\gamma)}.$$

Choosing $\sigma = 1$, the trivial representation of M , this simplifies to

$$\chi_1(\Gamma_\gamma) = \frac{\mathcal{O}_{b_\gamma}^M(f_1) \text{vol}(\Gamma_\gamma \backslash G_\gamma)}{l_{\gamma_0}}. \quad (7)$$

To complete the proof of the theorem we shall show that the orbital integral $\mathcal{O}_{b_\gamma}^M(f_1)$ is positive. In the case we are considering

$$b_\gamma = \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix},$$

where 1 denotes the identity matrix in $\text{SL}_2(\mathbb{R})$, hence it is central and $M_\gamma = M$ so we have simply that

$$\mathcal{O}_{b_\gamma}^M(f_1) = f_1(b_\gamma). \quad (8)$$

The group $\bar{M} \cong \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ is the connected component of $M \cong S(\text{SL}_2^\pm(\mathbb{R}) \times \text{SL}_2^\pm(\mathbb{R}))$. M has a maximal compact subgroup $K_M \cong O(2) \times O(2)$ and compact Cartan $T_M \cong \text{SO}(2) \times \text{SO}(2)$. We have that $T_M \bar{M} = \bar{M}$. Hence by Lemma 1.4, since f_1 is an Euler-Poincaré function for the trivial representation on M we have that $\bar{f}_1 = f_1|_{\bar{M}}$ is an Euler-Poincaré function for the trivial representation on \bar{M} .

Let g_1, h_1 be Euler-Poincaré functions of the trivial representation on $\mathrm{SL}_2(\mathbb{R})$. By Lemma 1.5 the function $\tilde{f}_1 = g_1 h_1$ is an Euler-Poincaré function of the trivial representation on \widehat{M} .

We recall that

$$b_\gamma = \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix},$$

which is central in M and deduce from Lemma 1.6 that

$$f_1(b_\gamma) = \bar{f}_1(b_\gamma) = \tilde{f}_1(b_\gamma) = g_1(\pm 1)h_1(\pm 1).$$

From Lemmas 1.6 and 1.7 it follows that

$$g_1(1) = g_1(-1) = h_1(1) = h_1(-1) \in \mathbb{R}$$

and so $f_1(b_\gamma)$ is positive. □

1.5 The unitary dual of K_M

Let $K_M = K \cap M \cong \mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(2))$. We shall need in the proof of later results to know the unitary dual $\widehat{K_M}$ of K_M , which is given in the following proposition.

First we define the following one dimensional representations of $\mathrm{SO}(2)$ and $\mathrm{SO}(2) \times \mathrm{SO}(2)$.

$$\begin{aligned} \varepsilon_l R(\theta) &= e^{il\theta}, \quad \text{for all } l \in \mathbb{Z}, \\ \varepsilon_{l,k} \begin{pmatrix} R(\theta) & \\ & R(\eta) \end{pmatrix} &= e^{i(l\theta + k\eta)}, \quad \text{for all } l, k \in \mathbb{Z}, \end{aligned}$$

where

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in \mathrm{SO}(2).$$

Note that ε_0 and $\varepsilon_{0,0}$ are the trivial representation of their respective groups.

Proposition 1.11 *For $l, k \in \mathbb{Z}$ not both zero there are unique representations $\delta_{l,k}$ of K_M on \mathbb{C}^2 with*

$$\delta_{l,k}|_{\mathrm{SO}(2) \times \mathrm{SO}(2)} = \varepsilon_{l,k} \oplus \varepsilon_{-l,-k},$$

and

$$\delta_{l,k} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} (z_1, z_2) = (z_2, z_1).$$

We can also define the representation δ of K_M on \mathbb{C} by

$$\delta(X, Y)(z) = (\det X)z = (\det Y)z.$$

We have that $\widehat{K_M} = \{\mathrm{triv}, \delta, \delta_{l,k} : l, k \in \mathbb{Z} \text{ not both zero}\}$.

Proof: We have seen that $\widehat{\mathrm{SO}(2)} = \{\varepsilon_l : l \in \mathbb{Z}\}$. In general, for two locally compact groups H and K , the map $(\sigma, \tau) \mapsto \sigma \otimes \tau$ defines an isomorphism $\widehat{H} \times \widehat{K} \cong \widehat{H \times K}$. Thus the map from $\widehat{\mathrm{SO}(2)} \times \widehat{\mathrm{SO}(2)}$ to $\widehat{\mathrm{SO}(2) \times \mathrm{SO}(2)}$ given by $(\tau_1, \tau_2) \mapsto \tau_1 \otimes \tau_2$ is an isomorphism. Hence we have that $\mathrm{SO}(2) \times \mathrm{SO}(2)$ is the set

$$\{\varepsilon_l \otimes \varepsilon_k \mid l, k \in \mathbb{Z}\} = \{\varepsilon_{l,k} \mid l, k \in \mathbb{Z}\}.$$

Let

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

and

$$R(\theta, \eta) = \begin{pmatrix} R(\theta) & \\ & R(\eta) \end{pmatrix}.$$

For $l, k \in \mathbb{Z}$ we note that

$$\delta_{l,k}(T)\delta_{l,k}(T)(z_1, z_2) = (z_1, z_2) = \delta_{l,k}(T^2)(z_1, z_2)$$

and

$$\begin{aligned} \delta_{l,k}(T)\delta_{l,k}(R(\theta, \eta))\delta_{l,k}(T)(z_1, z_2) &= (e^{-i(l\theta+k\eta)}z_1, e^{i(l\theta+k\eta)}z_2) \\ &= \delta_l(R(-\theta, -\eta))(z_1, z_2) \\ &= \delta_l(TR(\theta, \eta)T)(z_1, z_2), \end{aligned}$$

so $\delta_{l,k}$ is indeed a representation. We shall show that the representations given in the proposition are in fact the only irreducible unitary representations of K_M .

Let $\tau \in \widehat{K_M}$. Then by [26], Theorem 1.12(d) the representation τ restricted to $\mathrm{SO}(2) \times \mathrm{SO}(2)$ is a direct sum of irreducible representations, that is

$$\tau|_{\mathrm{SO}(2) \times \mathrm{SO}(2)} = \varepsilon_{l_1, k_1} \oplus \varepsilon_{l_2, k_2} \oplus \cdots \oplus \varepsilon_{l_n, k_n},$$

for some $l_1, \dots, l_n, k_1, \dots, k_n \in \mathbb{Z}$. Let $v \in \tau$ be an $(\mathrm{SO}(2) \times \mathrm{SO}(2))$ -eigenvector with eigenvalue $e^{i(l_1\theta+k_1\eta)}$. Then the equation

$$TR(-\theta, -\eta) = R(\theta, \eta)T$$

tells us that $\tau(T)v$ is also an $\mathrm{SO}(2)$ -eigenvector, with eigenvalue $e^{-i(l_1\theta+k_1\eta)}$. If $\tau(T)v$ is a scalar multiple of v then $l_1 = k_1 = 0$ and the equation $T^2 = I$ tells us that $\tau = \mathrm{triv}$ or δ . Otherwise $\tau|_{\mathrm{SO}(2) \times \mathrm{SO}(2)} = \varepsilon_{l_1, k_1} \oplus \varepsilon_{-l_1, -k_1}$ and $\tau = \delta_{l_1, k_1}$.

For $l = k = 0$ we have $\delta_{0,0} \cong \mathrm{triv} \oplus \delta$. For all other values of l and k the representation $\delta_{l,k}$ is irreducible. Also, for $l, k \in \mathbb{Z}$ we have that $\delta_{l,k}$ is unitarily equivalent to $\delta_{-l, -k}$. There are no other equivalences between the representations $\delta_{l,k}$.

The proposition follows. \square

1.6 Infinitesimal characters

Let G be a connected reductive group with maximal compact subgroup K , real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ and complexified Lie algebra \mathfrak{g} . Let π be a representation of G . Let π_K be the (\mathfrak{g}, K) -module of K -finite vectors in π .

Suppose that π is an admissible representation of G such that $\pi(Z)$ acts as a scalar on π_K for all Z in the centre \mathfrak{z}_G of the universal enveloping algebra of \mathfrak{g} . In particular this condition is satisfied when π is irreducible admissible. Let then \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} and let $\Lambda_\pi \in \mathfrak{h}^*$ be a representative of the infinitesimal character of π .

Let $G = \mathrm{SL}_4(\mathbb{R})$ and the subgroups K and $P = MAN$ be as above. Let \mathfrak{h} be the diagonal subalgebra of \mathfrak{g} . In this case the Weyl group $W(\mathfrak{g}, \mathfrak{h})$ acts on \mathfrak{h} by interchanging elements of the leading diagonal. We shall see in the next section that the analytic properties of the zeta functions considered there are related to the infinitesimal characters of elements of \hat{G} . The following proposition gives us information about these infinitesimal characters which will be required in the following section.

Let ρ_P be the half-sum of the positive roots of the system $(\mathfrak{g}, \mathfrak{a})$, where \mathfrak{a} is the complexified Lie algebra of A , so that

$$\rho_P \begin{pmatrix} a & & & \\ & a & & \\ & & -a & \\ & & & -a \end{pmatrix} = 4a.$$

Let σ be a finite dimensional virtual representation of M , whose K_M types are all contained in the set $\{\mathrm{triv}, \delta, \delta_{l,k} : l, k \in \{0, 2\}\}$. Let \hat{M}_σ be the subset of all $\xi \in \hat{M}$ such that $\mathrm{tr} \xi(f_\sigma) = 0$ for all Euler-Poincaré functions f_σ for σ on M . (Note that the value of $\mathrm{tr} \xi(f_\sigma) = 0$ depends only on ξ and σ and not on the choice of Euler-Poincaré function f_σ .) Let \hat{G}_σ be the set of all elements of \hat{G} except for: the trivial representation; representations induced from parabolic subgroups other than $P = MAN$ and representations induced from $\xi \in \hat{M}_\sigma$. We define an order on the real dual space of \mathfrak{a} by $\lambda > \mu$ if and only if $(\lambda - \mu) = t\rho_P$ for some $t > 0$.

Proposition 1.12 *Let σ be a finite dimensional virtual representation of M and let $\pi \in \hat{G}_\sigma$. Then the infinitesimal character Λ_π of π satisfies*

$$\operatorname{Re}(w\Lambda_\pi)|_{\mathfrak{a}} \geq -\frac{1}{2}\rho_P \quad \text{or} \quad -\rho_P \geq \operatorname{Re}(w\Lambda_\pi)|_{\mathfrak{a}}$$

for every $w \in W(\mathfrak{g}, \mathfrak{h})$.

Proof: We shall prove the proposition by considering different cases in turn. By Theorem 1.2 we have the following cases to consider. The case when π is a principal series representation induced from $P = MAN$; the case when π is a complementary series representation induced from $P = MAN$; the case when π is a limit of complementary series representation and the case when π is one of the representations π_m for $m \in \mathbb{N}$.

First we consider the principal series representations. Let $\pi = \pi_{\xi, \nu} = \operatorname{Ind}_P^G(\xi \otimes \nu)$ be induced from P , where ξ is an irreducible, unitary representation of $M \cong \operatorname{S}(\operatorname{SL}_2^\pm(\mathbb{R}) \times \operatorname{SL}_2^\pm(\mathbb{R}))$ and $\nu \in \mathfrak{a}^*$ is imaginary (ie. $\nu \in i\mathfrak{a}_{\mathbb{R}}^*$, where $\mathfrak{a}_{\mathbb{R}}^*$ is the real dual space of \mathfrak{a}).

Let ξ' be an irreducible subspace of $\xi|_{\operatorname{SL}_2(\mathbb{R}) \times \operatorname{SL}_2(\mathbb{R})}$. Then ξ' has infinitesimal character $\Lambda_{\xi'}$ and $\Lambda_\xi = \Lambda_{\xi'}$. We have that $\xi' \cong \xi_1 \otimes \xi_2$ where ξ_1 and ξ_2 are irreducible unitary representations of $\operatorname{SL}_2(\mathbb{R})$.

To limit the possibilities for ξ_1 and ξ_2 that we need to consider we use the double induction formula (see [26], (7.4)).

Lemma 1.13 *(Double induction formula)*

Let $M_\diamond A_\diamond N_\diamond$ be a parabolic subgroup of M , so that $M_\diamond(A_\diamond A)(N_\diamond N)$ is a parabolic subgroup of G . If σ_\diamond is a unitary representation of M_\diamond and $\nu_\diamond \in \mathfrak{a}_\diamond^* = (\operatorname{Lie}_{\mathbb{C}} A_\diamond)^*$, $\nu \in \mathfrak{a}^*$, then there is a canonical equivalence of representations

$$\operatorname{Ind}_{MAN}^G \left(\operatorname{Ind}_{M_\diamond A_\diamond N_\diamond}^M (\sigma_\diamond \otimes \nu_\diamond) \otimes \nu \right) \cong \operatorname{Ind}_{M_\diamond(A_\diamond A)(N_\diamond N)}^G (\sigma_\diamond \otimes (\nu_\diamond + \nu)).$$

□

We may assume that neither ξ_1 nor ξ_2 are induced since then, by the double induction formula, we would have the case that π was induced from a parabolic other than $P = MAN$, which was excluded. By Theorem 1.1, the remaining possibilities for ξ_1 and ξ_2 are the trivial representation and the discrete series and limit of discrete series representations.

Let Λ_π be the infinitesimal character of π and Λ_{ξ_i} be the infinitesimal character of ξ_i , then $\Lambda_\xi = \Lambda_{\xi_1} + \Lambda_{\xi_2}$. Recall that \mathfrak{h} is the diagonal subalgebra of \mathfrak{g} . We lift Λ_ξ and ν to \mathfrak{h} by defining Λ_ξ to be zero on \mathfrak{a} and ν to be zero on the diagonal elements of \mathfrak{m} , so that $\Lambda_\pi = \Lambda_\xi + \nu$ ([26], Proposition 8.22). The Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$ acts on Λ_π . Let $w \in W$, we will show that either $\operatorname{tr} \xi(f_\sigma) = 0$ or

$$\operatorname{Re}(w\Lambda_\pi|_{\mathfrak{a}}) \geq -\frac{1}{2}\rho_P$$

or

$$-\rho_P \geq \operatorname{Re}(w\Lambda_\pi|_{\mathfrak{a}}).$$

First we take ξ_1 and ξ_2 to be the trivial representation. This gives us that $\Lambda_\xi(\operatorname{diag}(s, -s, t, -t)) = s + t$. If $\nu(\operatorname{diag}(a, a, -a, -a)) = \alpha a$, $\alpha \in i\mathbb{R}$, then

$$\begin{aligned} & \Lambda_\pi(\operatorname{diag}(a, b, c, -a - b - c)) \\ &= \Lambda_\xi(\operatorname{diag}(\frac{a-b}{2}, \frac{b-a}{2}, c + \frac{a+b}{2}, -c - \frac{a+b}{2})) + \alpha \frac{a+b}{2} \\ &= \frac{a-b}{2} + \left(c + \frac{a+b}{2} \right) + \alpha \frac{a+b}{2} \\ &= a + c + \frac{\alpha}{2}(a+b). \end{aligned} \tag{9}$$

If we let $w = 1$ then from above we see that $\operatorname{Re}(\Lambda_\pi|_{\mathfrak{a}}) = 0$. If we take w to be the transposition interchanging b and c we get

$$w\Lambda_\pi(\operatorname{diag}(a, b, c, -a - b - c)) = a + b + \frac{\alpha}{2}(a + c),$$

the real part of which when restricted to \mathfrak{a} gives $\frac{1}{2}\rho_P$. All other Weyl group elements are dealt with similarly and we see that $-\frac{1}{2}\rho_P \leq \operatorname{Re}(\Lambda_\pi|_{\mathfrak{a}}) \leq \frac{1}{2}\rho_P$ for all $w \in W$.

It remains to consider the case when either or both of ξ_1 and ξ_2 are a discrete series representation or limit of discrete series representation with parameter m_i . We set $\mathcal{D}_0^+ = \mathcal{D}_0^- = \text{triv}$ and let $\xi_i = \mathcal{D}_{m_i}^+$ or $\xi_i = \mathcal{D}_{m_i}^-$, where $m_i \geq 0$, and m_1 and m_2 are not both zero. From (4) we get

$$\operatorname{tr} \xi(f_\sigma) = \sum_{p=0}^{\dim \mathfrak{p}_M} (-1)^p \dim \left(V_\xi \otimes \bigwedge^p \mathfrak{p}_M \otimes V_\sigma \right)^{K_M}. \quad (10)$$

We shall use Proposition 1.11 to examine the K_M types to limit the possibilities for m_1, m_2 for which $\operatorname{tr} \xi(f_\sigma) \neq 0$.

Lemma 1.14 *For m_1, m_2 both non-zero we have the following isomorphism of K_M -modules:*

$$V_\xi \cong \bigoplus_{\substack{j \geq m_1, j \equiv m_1 \pmod{2} \\ k \geq m_2, k \equiv m_2 \pmod{2}}} \delta_{j,k} \oplus \delta_{-j,k}.$$

If $m_1 = 0$ we have

$$V_\xi \cong \bigoplus_{k \geq m_2, k \equiv m_2 \pmod{2}} \delta_{0,k},$$

and for $m_2 = 0$ we have

$$V_\xi \cong \bigoplus_{j \geq m_1, j \equiv m_1 \pmod{2}} \delta_{j,0}.$$

Proof: This follows from Lemma 1.8. □

Lemma 1.15 *We have the following isomorphisms of K_M -modules:*

$$\begin{aligned} \bigwedge^0 \mathfrak{p}_M &\cong \text{triv} \\ \bigwedge^1 \mathfrak{p}_M &\cong \delta_{2,0} \oplus \delta_{0,2} \\ \bigwedge^2 \mathfrak{p}_M &\cong \delta \oplus \delta \oplus \delta_{2,2} \oplus \delta_{2,-2} \\ \bigwedge^3 \mathfrak{p}_M &\cong \delta_{2,0} \oplus \delta_{0,2} \\ \bigwedge^4 \mathfrak{p}_M &\cong \text{triv}. \end{aligned}$$

Proof: K_M acts on \mathfrak{p}_M by the adjoint representation and we can compute

$$\mathfrak{p}_M \cong \delta_{2,0} \oplus \delta_{0,2}.$$

The other isomorphisms follow straightforwardly from this. □

Let $v = v_1 \otimes v_2 \otimes v_3 \in V_\xi \otimes \bigwedge^* \mathfrak{p}_M \otimes V_\sigma$, where the v_i 's all lie in a single K_M type of their respective representation spaces. Lemma 1.15 tells us that v_2 is in one of the following K_M types: triv , δ , $\delta_{2,0}$, $\delta_{0,2}$, $\delta_{2,2}$, $\delta_{2,-2}$ by our assumption on σ , the possibilities for the K_M type containing v_3 are also the same. It follows that $\operatorname{tr} \xi(f_\sigma)$ is non-zero only if $m_1, m_2 \in \{0, 2, 4\}$.

It follows from the exact sequences in the proof of Lemma 1.8 that $\mathcal{D}_m^+ \oplus \mathcal{D}_m^- \subset \mathcal{P}^{\pm, (m-1)\rho_1}$, where the index on \mathcal{P} is $+$ if m is even and $-$ if m is odd. The infinitesimal character of $\mathcal{P}^{\pm, (m-1)\rho_1}$ is simply $(m-1)\rho_1$ (see [26], Proposition 8.22), hence it follows that the infinitesimal characters of \mathcal{D}_m^+ and \mathcal{D}_m^- are both equal to $(m-1)\rho_1$. This gives us in the cases where $\operatorname{tr} \xi(f_\sigma)$ is non-zero:

$$\Lambda_\xi(\operatorname{diag}(s, -s, t, -t)) = (m_1 - 1)s + (m_2 - 1)t, \quad m_1, m_2 \in \{0, 2, 4\}. \quad (11)$$

By computing the action of the different Weyl group elements we can see that in all cases either $w\Lambda_\xi|_{\mathfrak{a}} \geq -\frac{1}{2}\rho_P$ or $w\Lambda_\xi|_{\mathfrak{a}} \leq -\rho_P$. Since $\Lambda_\pi = \Lambda_\xi + \nu$ and ν is imaginary we see that $\operatorname{Re}(\Lambda_\pi|_{\mathfrak{a}}) = \Lambda_\xi|_{\mathfrak{a}}$, so the claim follows.

The complementary series representations are dealt with similarly. We recall from Theorem 1.2 that the complementary series induced from P are $\pi = \text{Ind}_P^G(\bar{\pi}_m \otimes t\rho_P)$, for $m \in \mathbb{N}$ and $0 < t < \frac{1}{2}$. We may argue as above to find the possibilities for m such that $\text{tr } \bar{\pi}_m(f_\sigma) \neq 0$. In this way we see that there are two possibilities for m for which $\text{tr } \bar{\pi}_m(f_\sigma) \neq 0$, namely $m = 2$ and $m = 4$. In the first case, $w\Lambda_\pi|_{\mathfrak{a}} \geq -\frac{1}{2}\rho_P$, for all $w \in W(\mathfrak{g}, \mathfrak{h})$. In the second case we have

$$\Lambda_\pi(\text{diag}(a, b, c, -a - b - c)) = (3 + 2t)a + 2tb + 3c.$$

If $w \in W(\mathfrak{g}, \mathfrak{h})$ is the element which swaps the first and fourth diagonal entries then $w\Lambda_\pi|_{\mathfrak{a}} = -\frac{3}{2}\rho_P$. In all other cases we have $w\Lambda_\pi|_{\mathfrak{a}} \geq -\frac{1}{2}\rho_P$.

The limit of complementary series representations are closely related to the family of representations π_m , $m \in \mathbb{N}$, so we shall deal with them together.

For $m \in \mathbb{N}$, we denote by $\bar{\pi}_m$ the representation of M induced from the representation $\mathcal{D}_m^+ \otimes \mathcal{D}_m^+$ of $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$. For $m \in \mathbb{N}$ we have the limit of complementary series representation given as an irreducible unitary subrepresentation of $I_m = \text{Ind}_P^G(\bar{\pi}_m \otimes \frac{1}{2}\rho)$, which we will denote by π_m^c . The representations π_m are the Langlands quotients of the I_m .

Let Λ_{π_m} be the infinitesimal character of π_m and $\Lambda_{\pi_m^c}$ the infinitesimal character of π_m^c . Clearly $\Lambda_{\pi_m} = \Lambda_{\pi_m^c} = \Lambda_{\bar{\pi}_m} + \frac{1}{2}\rho_P$, so we need only consider Λ_{π_m} . The value of

$$\Lambda_{\pi_m} \begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & -a - b - c \end{pmatrix}$$

is equal to

$$\begin{aligned} & \Lambda_{\bar{\pi}_m} \begin{pmatrix} \frac{a-b}{2} & & & \\ & \frac{b-a}{2} & & \\ & & \frac{a+b+2c}{2} & \\ & & & \frac{a+b-2c}{2} \end{pmatrix} + \frac{1}{2}\rho_P \begin{pmatrix} \frac{a+b}{2} & & & \\ & \frac{a+b}{2} & & \\ & & -\frac{a+b}{2} & \\ & & & -\frac{a+b}{2} \end{pmatrix} \\ &= (m-1)\frac{a-b}{2} + (m-1)(c + \frac{a+b}{2}) + (a+b) \\ &= ma + b + (m-1)c. \end{aligned}$$

Restricting to \mathfrak{a} we get

$$\Lambda_{\pi_m} \begin{pmatrix} a & & & \\ & a & & \\ & & -a & \\ & & & -a \end{pmatrix} = 2a = \frac{1}{2}\rho_P.$$

The Weyl group W acts on Λ_{π_m} . Now for all elements $w \in W$ except one we get $w\Lambda_{\pi_m}|_{\mathfrak{a}} \geq -\frac{1}{2}\rho_P$ for all $m \geq 1$. The exception is w_1 which swaps the first and fourth diagonal entries. For this we have

$$w_1\Lambda_{\pi_m} \begin{pmatrix} a & & & \\ & a & & \\ & & -a & \\ & & & -a \end{pmatrix} = -2(m-1)a.$$

Thus $w_1\Lambda_{\pi_m}|_{\mathfrak{a}} \geq -\frac{1}{2}\rho_P$ if $m = 1, 2$ and if $m \geq 3$ then $w_1\Lambda_{\pi_m}|_{\mathfrak{a}} \leq -\rho_P$.

This completes the proof of the proposition. \square

2 Analysis of the Ruelle Zeta Function

Let $G = \text{SL}_4(\mathbb{R})$ and $\Gamma \subset G$ be discrete and cocompact. Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_4(\mathbb{R})$ and $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$ be respectively the Lie algebra and complexified Lie algebra of G . As in the previous section, all Haar measures will be normalised as in [19], §7. Recall that this normalisation depends on the choice of an invariant bilinear form b on \mathfrak{g} . Let b be the following multiple of the trace form on \mathfrak{g} :

$$b(X, Y) = 16 \text{tr}(XY). \tag{12}$$

We choose this normalisation in order to get the first zero of the Ruelle zeta function at the point $s = 1$ in Theorem 2.11 below. Let $K \subset G$ be the maximal compact subgroup $\mathrm{SO}(4)$. Let $\mathfrak{k}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ be its Lie algebra and let $\mathfrak{p}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ be the orthogonal space of $\mathfrak{k}_{\mathbb{R}}$ with respect to the form b . Then b is positive definite and $\mathrm{Ad}(K)$ -invariant on $\mathfrak{p}_{\mathbb{R}}$ and thus defines a G -invariant metric on $X = G/K$, the symmetric space attached to G .

Let

$$A = \left\{ \begin{pmatrix} a & & & \\ & a & & \\ & & a^{-1} & \\ & & & a^{-1} \end{pmatrix} : a > 0 \right\},$$

$$B = \begin{pmatrix} \mathrm{SO}(2) & \\ & \mathrm{SO}(2) \end{pmatrix}.$$

B is a compact Cartan subgroup of

$$M = \mathrm{S} \begin{pmatrix} \mathrm{SL}_2^{\pm}(\mathbb{R}) & \\ & \mathrm{SL}_2^{\pm}(\mathbb{R}) \end{pmatrix}$$

$$\cong \left\{ (x, y) \in \mathrm{Mat}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{R}) \mid \begin{array}{l} \det(x), \det(y) = \pm 1 \\ \det(x)\det(y) = 1 \end{array} \right\}.$$

We also define

$$N = \begin{pmatrix} I_2 & \mathrm{Mat}_2(\mathbb{R}) \\ 0 & I_2 \end{pmatrix} \text{ and } \bar{N} = \begin{pmatrix} I_2 & 0 \\ \mathrm{Mat}_2(\mathbb{R}) & I_2 \end{pmatrix},$$

and set $K_M = K \cap M$.

Let \mathfrak{m} denote the complexified Lie algebra of M and let $\mathfrak{m} = \mathfrak{k}_M \oplus \mathfrak{p}_M$ be its polar decomposition, where \mathfrak{k}_M is the complexified Lie algebra of $K_M = K \cap M$. Let \mathfrak{h} be the Cartan subalgebra of \mathfrak{g} consisting of all diagonal matrices, and let \mathfrak{a} , \mathfrak{n} and $\bar{\mathfrak{n}}$ be the complexified Lie algebras of A , N and \bar{N} respectively. Let P denote the parabolic with Langlands decomposition $P = MAN$ and \bar{P} the opposite parabolic with Langlands decomposition $\bar{P} = M\bar{A}\bar{N}$. Let ρ_P be the half-sum of the positive roots of the system $(\mathfrak{g}, \mathfrak{a})$, so that

$$\rho_P \begin{pmatrix} a & & & \\ & a & & \\ & & -a & \\ & & & -a \end{pmatrix} = 4a.$$

Let

$$H_1 = \frac{1}{8} \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in \mathfrak{a}_{\mathbb{R}}.$$

Then it follows that $b(H_1) = b(H_1, H_1) = 1$ and $\rho_P(H_1) = -\frac{1}{2}$. Let $A^- = \{\exp(tH_1) : t > 0\}$ be the negative Weyl chamber in A . Let $\mathcal{E}_P(\Gamma)$ be the set of all conjugacy classes $[\gamma]$ in Γ such that γ is conjugate in G to an element $a_{\gamma}b_{\gamma}$ of A^-B .

An element $\gamma \in \Gamma$ is called *primitive* if for $\delta \in \Gamma$ and $n \in \mathbb{N}$ the equation $\delta^n = \gamma$ implies that $n = 1$. Clearly primitivity is invariant under conjugacy, so we may view it as a property of conjugacy classes. Let $\mathcal{E}_P^p(\Gamma) \subset \mathcal{E}_P(\Gamma)$ be the subset of primitive classes.

For $[\gamma] \in \mathcal{E}_P(\Gamma)$ we define the length of γ to be $l_{\gamma} = b(\log a_{\gamma}, \log a_{\gamma})^{1/2}$. Let G_{γ} be the centraliser of γ in G , let $\Gamma_{\gamma} = \Gamma \cap G_{\gamma}$ be the centraliser of γ in Γ and let $\chi_1(\Gamma_{\gamma})$ be the first higher Euler characteristic as in the previous section.

2.1 The Selberg trace formula

Let \hat{G} be the unitary dual of G . The group G acts on the Hilbert space $L^2(\Gamma \backslash G)$ by translations from the right. Since $\Gamma \backslash G$ is compact this representation decomposes discretely:

$$L^2(\Gamma \backslash G) = \bigoplus_{\pi \in \hat{G}} N_{\Gamma}(\pi) \pi$$

with finite multiplicities $N_\Gamma(\pi)$, (see [17]).

Recall that for $\gamma \in \Gamma$ we denote by G_γ and Γ_γ the centraliser of γ in G and Γ respectively. For a function f on G , denote by $\mathcal{O}_\gamma(f)$ the orbital integral

$$\mathcal{O}_\gamma(f) = \int_{G/G_\gamma} f(x\gamma x^{-1}) dx.$$

The *Selberg trace formula* (see [40], Theorem 2.1) says that for suitable functions f on G the following identity holds:

$$\sum_{\pi \in \hat{G}} N_\Gamma(\pi) \operatorname{tr} \pi(f) = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_\gamma(f), \quad (13)$$

where the sum on the right is over all conjugacy classes in Γ . The set of suitable functions includes, but is not limited to, all $\dim G + 1$ times continuously differentiable functions of compact support on G . In fact, we shall need to extend the set of test functions for which the trace formula is valid.

Lemma 2.1 *Let $d \in \mathbb{N}$, $d \geq 16$, that is $d = 2d'$ for some $d' > \dim G/2$. Let f be a d -times differentiable function on G such that $Df \in L^1(G)$ for all left invariant differentiable operators D on G with complex coefficients and of degree $\leq d$. Then the trace formula is valid for f .*

Proof: This is Lemma 1.3 of [10] in the case $G = \operatorname{SL}_4(\mathbb{R})$. □

For $g \in G$ and V any complex vector space on which g acts linearly let $E(g|V) \subset \mathbb{R}_+$ be defined by

$$E(g|V) = \{|\mu| : \mu \text{ is an eigenvalue of } g \text{ on } V\}.$$

Let $\lambda_{\min}(g|V) = \min(E(g|V))$ and $\lambda_{\max}(g|V) = \max(E(g|V))$.

For $am \in AM$ define

$$\lambda(am) = \frac{\lambda_{\min}(a|\mathfrak{n})}{\lambda_{\max}(m|\mathfrak{g})^2},$$

where we are considering the adjoint action of G on \mathfrak{n} and \mathfrak{g} respectively. Define the set

$$(AM)^\sim = \{am \in AM : \lambda(am) < 1\}.$$

An element of G is said to be *elliptic* if it is conjugate to an element of a compact torus. Let M_{ell} denote the set of elliptic elements in M .

Lemma 2.2 *The set $(AM)^\sim$ has the following properties:*

- (1) $A^- M_{\text{ell}} \subset (AM)^\sim$;
- (2) $am \in (AM)^\sim \Rightarrow a \in A^-$;
- (3) $am, a'm' \in (AM)^\sim, g \in G$ with $a'm' = gamg^{-1} \Rightarrow a = a', g \in AM$.

Proof: See [8], Lemma 2.4. □

For the construction of our test function we shall need, for given $s \in \mathbb{C}$ and $j \in \mathbb{N}$, a smooth, conjugation invariant, j -times continuously differentiable function on AM , with support in $(AM)^\sim$. Further, we require that at each point $ab \in A^- B$ the function takes the value $l_a^{j+1} e^{-sl_a}$. In [8] a function g_s^j is constructed with these properties, with the one difference that there the positive Weyl chamber A^+ is used instead of the negative Weyl chamber A^- . With only very minor modification the construction in [8] will yield a function with our required properties, which we shall also call g_s^j .

Let $\eta : N \rightarrow \mathbb{R}$ be a smooth, non-negative function of compact support, which is invariant under conjugation by elements of K_M and satisfies

$$\int_N \eta(n) dn = 1.$$

Let $f : M \rightarrow \mathbb{C}$ be a smooth, compactly supported function, invariant under conjugation by K_M . Suppose further that the orbital integrals of f on M satisfy

$$\mathcal{O}_m^M(f) = \int_{M/M_m} f(xmx^{-1}) dx = 0$$

whenever m is not conjugate to an element of B , where M_m denotes the centraliser of m in M . The group AM acts on \mathfrak{n} according to the adjoint representation.

Given these data we define $\Phi = \Phi_{f,j,s} : G \rightarrow \mathbb{C}$ by

$$\Phi(knam(kn)^{-1}) = \eta(n)f(m) \frac{g_s^j(am)}{\det(1 - (am)^{-1}|\bar{\mathfrak{n}})}, \quad (14)$$

for $k \in K, n \in N, am \in AM$.

To see that Φ is well defined we recall that, by the decomposition $G = KNAM$, every $g \in G$ which is conjugate to an element of $(AM)^\sim$ can be written in the form $knam(kn)^{-1}$. By the properties of $(AM)^\sim$ we see that two of these representations can only differ by an element of K_M . The components of the function Φ are all invariant under conjugation by K_M , and we note that $\det(1 - (am)^{-1}|\bar{\mathfrak{n}}) \neq 0$ for all $am \in (AM)^\sim$, so we can see that Φ is well-defined.

Proposition 2.3 *The function Φ is $(j - 14)$ -times continuously differentiable. For j and $\operatorname{Re}(s)$ large enough it goes into the trace formula and we have:*

$$\sum_{\pi \in \hat{G}} N_\Gamma(\pi) \operatorname{tr} \pi(\Phi) = \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_{b_\gamma}^M(f) \frac{l_{a_\gamma}^{j+1} e^{-sl_{a_\gamma}}}{\det(1 - (a_\gamma b_\gamma)^{-1}|\bar{\mathfrak{n}})}. \quad (15)$$

Proof: Noting that $2 \dim \mathfrak{n} + \dim \mathfrak{k} = 14$ we see that this follows from Proposition 2.5 of [8]. This proposition was proved in the case that the function f is an Euler-Poincaré function for some finite dimensional representation of M . However the only properties of Euler-Poincaré functions used in the proof are those given above for f , namely that it is smooth, of compact support, invariant under conjugation by K , and the orbital integrals satisfy the given condition. \square

2.2 The Selberg zeta function

An element $g \in G$ is said to be *weakly neat* if the adjoint $\operatorname{Ad}(g)$ has no non-trivial roots of unity as eigenvalues. A subgroup H of G is said to be weakly neat if every element of H is weakly neat. Let $[\gamma] \in \mathcal{E}_P^p(\Gamma)$ so that γ is conjugate in G to $a_\gamma b_\gamma \in A^- B$. We want to know which roots of unity can occur as eigenvalues of $\operatorname{Ad}(\gamma)$. These are the same as the roots of unity which occur as eigenvalues of $\operatorname{Ad}(a_\gamma b_\gamma)$. If

$$b_\gamma = \begin{pmatrix} R(\theta) & \\ & R(\phi) \end{pmatrix},$$

where

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

then the eigenvalues of $\operatorname{Ad}(a_\gamma b_\gamma)$ are $e^{\pm 2i\theta}$ and $e^{\pm 2i\phi}$. Define the sets

$$R_\theta = \{n \in \mathbb{N} : n\theta \in \pi\mathbb{Z}\} \quad \text{and} \quad R_\phi = \{n \in \mathbb{N} : n\phi \in \pi\mathbb{Z}\}$$

and

$$R_\gamma = \{\min R_\theta, \min R_\phi\}.$$

Then R_γ contains either 0, 1 or 2 elements. We can see that γ weakly neat is equivalent to $R_\gamma = \emptyset$. For $\gamma \in \mathcal{E}_P(\Gamma)$, where $\gamma = \gamma_0^n$ for γ_0 primitive, the value of $\chi_1(\Gamma_\gamma)$ depends on whether or not $n \in R_{\gamma_0}$.

For $I \subset R_\gamma$ with $I \neq \emptyset$ define n_I to be the least common multiple of the elements of I and set $n_\emptyset = 1$. Further, define

$$\chi_I(\gamma) = \frac{(-1)^{|I|}}{n_I} \sum_{J \subset I} (-1)^{|J|} \chi_1(\Gamma_{\gamma^{n_J}}).$$

Let $z \in \mathbb{C} \setminus \{0\}$ and $q \in \mathbb{Q}$. We define z^q to be equal to $e^{q \log z}$, where we take the branch of the logarithm with imaginary part in the interval $(-\pi, \pi]$.

For any finite-dimensional virtual representation σ of M we define, for $\operatorname{Re}(s)$ large, the *generalised Selberg zeta function*

$$Z_{P,\sigma}(s) = \prod_{[\gamma] \in \mathcal{E}_P^p(\Gamma)} \prod_{n \geq 0} \prod_{I \subset R_\gamma} \det(1 - e^{-sn_I l_\gamma} \gamma^{n_I} |V_\sigma \otimes S^n(\mathfrak{n}))^{\chi_I(\gamma)}, \quad (16)$$

where $S^n(\mathfrak{n})$ denotes the n^{th} symmetric power of the space \mathfrak{n} and γ acts on $V_\sigma \otimes S^n(\mathfrak{n})$ via $\sigma(b_\gamma) \otimes \text{Ad}^n(a_\gamma b_\gamma)$. In the case that Γ is weakly neat this simplifies to

$$Z_{P,\sigma}(s) = \prod_{[\gamma] \in \mathcal{E}_P^b(\Gamma)} \prod_{n \geq 0} \det(1 - e^{-sl_\gamma} \gamma | V_\sigma \otimes S^n(\mathfrak{n}))^{\chi_1(\Gamma_\gamma)},$$

(see [8]).

Let $\pi \in \hat{G}$. The Lie algebra \mathfrak{n} acts on π_K and we denote by $H^\bullet(\mathfrak{n}, \pi_K)$ (resp. $H_\bullet(\mathfrak{n}, \pi_K)$) the corresponding Lie algebra cohomology (resp. homology) (see [3],[4]). We have the following isomorphism of AM -modules (see [20], p57):

$$H_p(\mathfrak{n}, \pi_K) \cong H^{4-p}(\mathfrak{n}, \pi_K) \otimes \bigwedge^4 \mathfrak{n}. \quad (17)$$

For $\lambda \in \mathfrak{a}^*$ let

$$m_\lambda(\pi) = \sum_{p,q \geq 0} (-1)^{p+q} \dim \left(H^q(\mathfrak{n}, \pi_K)^\lambda \otimes \bigwedge^p \mathfrak{p}_M \otimes V_\sigma \right)^{K_M}, \quad (18)$$

where for an \mathfrak{a} -module V and $\lambda \in \mathfrak{a}^*$, V^λ denotes the generalised λ -eigenspace

$$\{v \in V \mid \exists n \in \mathbb{N} \text{ such that } (a - \lambda(a))^n v = 0 \ \forall a \in \mathfrak{a}\}$$

and the superscript K_M denotes the subspace of K_M invariants.

We say that an admissible representation π of a linear connected reductive group G' has a *global character* $\Theta = \Theta_\pi^{G'}$ if for all smooth, compactly supported functions f on G' the operator $\pi(f)$ is of trace class and $\Theta_\pi^{G'}$ is a locally L^2 function on G' satisfying

$$\text{tr } \pi(f) = \int_{G'} \Theta_\pi^{G'}(g) f(g) dg$$

for each such f . By [26], Theorem 10.2, every irreducible admissible, and in particular every irreducible unitary, representation of G' has a global character.

Theorem 2.4 *Let σ be a finite dimensional virtual representation of M . The function $Z_{P,\sigma}(s)$ extends to a meromorphic function on the whole of \mathbb{C} . For $\lambda \in \mathfrak{a}^*$, the vanishing order of $Z_{P,\sigma}(s)$ at the point $s = \lambda(H_1)$ is*

$$\sum_{\pi \in \hat{G}} N_\Gamma(\pi) m_\lambda(\pi). \quad (19)$$

Further, all the poles and zeros of $Z_{P,\sigma}(s)$ lie in $\mathbb{R} \cup (\frac{1}{2} + i\mathbb{R})$.

Proof: The analogue of this theorem in the case that G has trivial centre and Γ is weakly neat is proved in [8], Theorem 2.1. If G has trivial centre then Γ weakly neat implies Γ torsion free, since non weakly neat torsion elements must be central. With a few modifications the proof of [8], Theorem 2.1 becomes valid in our case also. In fact the assumption that Γ is weakly neat is used in two places. We sketch the proof here, pointing out the necessary modifications for it to be valid in our case.

The theorem is proved by setting $f = f_\sigma$ in the test function $\Phi = \Phi_{j,s} = \Phi_{\sigma,j,s}$ defined in (14), where f_σ is an Euler-Poincaré function for σ on M . We then get that for $j \in \mathbb{N}$ and $\text{Re}(s)$ sufficiently large, the right hand side of (15) is equal to

$$\sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \text{vol}(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_{b_\gamma}^M(f_\sigma) \frac{l_\gamma^{j+1} e^{-sl_\gamma}}{\det(1 - (a_\gamma b_\gamma)^{-1} | \bar{\mathfrak{n}})}. \quad (20)$$

In [8] the following equation from [7], proved under the assumption that Γ is torsion free, is used:

$$\chi_1(\Gamma_\gamma) = \frac{C_\gamma \text{vol}(\Gamma_\gamma \backslash G_\gamma)}{l_{\gamma_0}}, \quad (21)$$

where C_γ is an explicit constant depending only on γ , and γ_0 denotes the primitive geodesic underlying γ . We have shown in Lemma 1.9 that (21) holds for all $\gamma \in \mathcal{E}_P(\Gamma)$ in our case also. From [8], Proposition 1.4 we take the equation

$$\mathcal{O}_{b_\gamma}^M(f_\sigma) = C_\gamma \text{tr } \sigma(b_\gamma),$$

which, together with (21) gives us

$$\mathcal{O}_{b_\gamma}^M(f_\sigma) \text{vol}(\Gamma_\gamma \backslash G_\gamma) = l_{\gamma_0} \text{tr } \sigma(b_\gamma) \chi_1(\Gamma_\gamma).$$

Substituting this into (20) we get

$$\sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} l_{\gamma_0} \text{tr } \sigma(b_\gamma) \chi_1(\Gamma_\gamma) \frac{l_\gamma^{j+1} e^{-sl_\gamma}}{\det(1 - (a_\gamma b_\gamma)^{-1} | \bar{\mathbf{n}})}.$$

We claim that this is equal to

$$(-1)^{j+1} \left(\frac{\partial}{\partial s} \right)^{j+2} \log Z_{P,\sigma}(s).$$

Indeed

$$\log Z_{P,\sigma}(s) = - \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \sum_{m=1}^{\infty} \sum_{I \subset R_\gamma} \chi_I(\gamma) \frac{e^{-sm l_\gamma}}{m} \text{tr } \sigma(b_\gamma^m) \sum_{n \geq 0} \text{tr } ((a_\gamma b_\gamma)^m | S^n(\mathbf{n})).$$

We note that

$$\begin{aligned} \sum_{n \geq 0} \text{tr } ((a_\gamma b_\gamma)^m | S^n(\mathbf{n})) &= \det(1 - (a_\gamma b_\gamma)^m | \mathbf{n})^{-1} \\ &= \det(1 - (a_\gamma b_\gamma)^{-m} | \bar{\mathbf{n}})^{-1} \end{aligned}$$

and the claimed equality follows.

In [8], since Γ is weakly neat it follows that $G_{\gamma^n} = G_\gamma$ for all $\gamma \in \Gamma$ and $n \in \mathbb{N}$. In [14] it is shown that $X_\gamma \cong \Gamma_\gamma \backslash G_\gamma / K_\gamma$ and so it follows that for all $n \in \mathbb{N}$ we have $X_{\gamma^n} \cong X_\gamma$ and hence $\chi_1(\Gamma_{\gamma^n}) = \chi_1(\Gamma_\gamma)$. Thus the above equality involving the logarithmic derivative of $Z_{P,\sigma}(s)$ is derived with the simpler Euler product expansion for $Z_{P,\sigma}(s)$ given above.

In our case we may have an element $[\gamma] \in \mathcal{E}_P^P(\Gamma)$ with a non-trivial root of unity as an eigenvalue. For such a γ we have $\chi_1(\Gamma_{\gamma^n}) \neq \chi_1(\Gamma_\gamma)$ for $n \in R_\gamma$. For this reason we have had to introduce the more complicated Euler product expansion for $Z_{P,\sigma}(s)$ so that the above equality involving the logarithmic derivative of $Z_{P,\sigma}(s)$ still holds.

On page 909 of [8] it is shown that

$$\text{tr } \pi(\Phi_s) = \int_{MA^-} f_\sigma(m) \Theta_{H^\bullet(\mathbf{n}, \pi_K)}^{MA}(ma) dm g_s^j(a) da. \quad (22)$$

Using the property (4) of f_σ we get

$$\begin{aligned} \text{tr } \pi(\Phi_s) &= \int_{A^-} \text{tr} \left(a \left| \left(H^\bullet(\mathbf{n}, \pi_K) \otimes \bigwedge^* \mathfrak{p}_M \otimes V_\sigma \right)^{K_M} \right| \right) l_a^{j+1} e^{-sl_a} da \\ &= \int_0^\infty \sum_{\lambda \in \mathfrak{a}^*} m_\lambda(\pi) e^{(\lambda(H_1) - s)t} t^{j+1} dt \\ &= (-1)^{j+1} \left(\frac{\partial}{\partial s} \right)^{j+1} \sum_{\lambda \in \mathfrak{a}^*} m_\lambda(\pi) \frac{1}{s - \lambda(H_1)}. \end{aligned} \quad (23)$$

Proposition 2.3 then gives us that

$$\left(\frac{\partial}{\partial s} \right)^{j+2} \log Z_{P,\sigma}(s) = \sum_{\pi \in \hat{G}} N_\Gamma(\pi) \left(\frac{\partial}{\partial s} \right)^{j+1} \sum_{\lambda \in \mathfrak{a}^*} m_\lambda(\pi) \frac{1}{s - \lambda(H_1)}, \quad (24)$$

from which it follows that the vanishing-order of $Z_{P,\sigma}(s)$ at $s = \lambda(H_1)$ is

$$\sum_{\pi \in \hat{G}} N_\Gamma(\pi) m_\lambda(\pi).$$

Two further comments are in order. First, in [8] the positive Weyl chamber $A^+ = \{\exp(tH_1) : t < 0\}$ in A is considered, where we have instead considered the negative chamber A^- . For this reason we have interchanged the positions of the Lie algebras \mathfrak{n} and $\bar{\mathfrak{n}}$ from the way they are used in [8]. This is easily seen to give a precisely equivalent result.

Secondly, the results of [8] are stated for a finite dimensional representation σ of M . By linearity the results extend in a straightforward way to the case where σ is a virtual representation, which we use here. \square

2.3 A functional equation for $Z_{P,\sigma}(s)$

Proposition 2.5 *For $\lambda \in \mathfrak{a}^*$ let $\|\lambda\|$ be the norm given by the form b in (12). There are $m_1 \in \mathbb{N}$ and $C > 0$ such that for every $\pi \in \hat{G}$ and every $\lambda \in \mathfrak{a}^*$ we have*

$$\left| \sum_{q=0}^4 (-1)^q \dim(H^q(\mathfrak{n}, \pi_K)^\lambda) \right| \leq C(1 + \|\lambda\|)^{m_1}.$$

Further, let S denote the set of all pairs $(\pi, \lambda) \in \hat{G} \times \mathfrak{a}^*$ such that

$$\sum_{q=0}^4 (-1)^q \dim(H^q(\mathfrak{n}, \pi_K)^\lambda) \neq 0.$$

Then there is $m_2 \in \mathbb{N}$ such that

$$\sum_{(\pi, \lambda) \in S} \frac{N_\Gamma(\pi)}{(1 + \|\lambda\|)^{m_2}} < \infty.$$

Proof: These results follow from [11], Proposition 2.4 and Lemma 2.7. \square

An entire function f is said to be of *finite order* if there is a positive constant a and an $r > 0$ such that $|f(z)| < \exp(|z|^a)$ for $|z| > r$. If f is of finite order then the *order of f* is defined to be the infimum of such a 's.

It is well known that the classical Selberg zeta function is an entire function of order two (see [37]). Our next lemma gives an analogous result for the generalised Selberg zeta function we are considering here.

Let σ be a finite dimensional virtual representation of M . Theorem 2.4 tells us that $Z_{P,\sigma}(s)$ is meromorphic and hence it may be written as the quotient of two entire functions:

$$Z_{P,\sigma}(s) = \frac{Z_1(s)}{Z_2(s)},$$

where the zeros of $Z_1(s)$ correspond to the zeros of $Z_{P,\sigma}(s)$ and the zeros of $Z_2(s)$ correspond to the poles of $Z_{P,\sigma}(s)$. The orders of the zeros of $Z_1(s)$ (resp. $Z_2(s)$) equal the orders of the corresponding zeros (resp. poles) of $Z_{P,\sigma}(s)$. The functions $Z_1(s)$ and $Z_2(s)$ are not uniquely determined, but clearly their respective sets of zeros, together with the orders of the zeros, are. For $i = 1, 2$ let R_i denote the set of zeros of $Z_i(s)$ counted with multiplicity.

Lemma 2.6 *There exist $Z_i(s)$, for $i = 1, 2$, with the above properties, which are, in addition, of finite order.*

Proof: Let $Z_1(s)$, $Z_2(s)$ be as above. We shall show that we may take them to be of finite order.

For $\pi \in \hat{G}$ let $\Lambda(\pi)$ be the set of all $\lambda \in \mathfrak{a}^*$ with $\lambda \neq 0$ and $m_\lambda(\pi) \neq 0$. Let $\hat{G}(\Gamma)$ be the set of $\pi \in \hat{G}$ such that $N_\Gamma(\pi) \neq 0$ and let S denote the set of all pairs (π, λ) such that $\pi \in \hat{G}(\Gamma)$ and $\lambda \in \Lambda(\pi)$. For $\lambda \in \mathfrak{a}^*$ let $\|\lambda\|$ be the norm given by the form b in (12).

The expression (19) tells us that $s \neq 0$ is a zero or pole of $Z_P(s)$ if and only if $s = \lambda(H_1)$ for some $\lambda \in \mathfrak{a}^*$, for which there exists $\pi \in \hat{G}(\Gamma)$ such that $(\pi, \lambda) \in S$.

Since the function $Z_{P,\sigma}(s)$ is meromorphic and non-zero, it follows that there exists $c > 0$ such that

$$|\lambda(H_1)| \geq c(1 + \|\lambda\|) \tag{25}$$

for all λ such that $(\pi, \lambda) \in S$ for some π .

By the definition of $m_\lambda(\pi)$ we deduce immediately from Proposition 2.5 that there exist $m_1 \in \mathbb{N}$ and $C > 0$ such that for every $\pi \in \hat{G}$ and every $\lambda \in \mathfrak{a}^*$ we have

$$|m_\lambda(\pi)| \leq C(1 + \|\lambda\|)^{m_1}, \tag{26}$$

and that there exists $m_2 \in \mathbb{N}$ such that

$$\sum_{(\pi, \lambda) \in S} \frac{N_\Gamma(\pi)}{(1 + \|\lambda\|)^{m_2}} < \infty. \tag{27}$$

Let $k = m_1 + m_2$. Then, for $(\pi, \lambda) \in S$, by (25) and (26) we have,

$$\begin{aligned} \left| \frac{m_\lambda(\pi)}{\lambda(H_1)^k} \right| &\leq \frac{1}{c^k} \cdot \frac{|m_\lambda(\pi)|}{(1 + \|\lambda\|)^k} \\ &\leq \frac{C}{c^k} \cdot \frac{1}{(1 + \|\lambda\|)^{m_2}}. \end{aligned}$$

It then follows from (19) and (27) that, for $i = 1, 2$,

$$\sum_{\rho \in R_i \setminus \{0\}} \frac{1}{|\rho|^k} \leq \sum_{(\pi, \lambda) \in S} \frac{N_\Gamma(\pi) |m_\lambda(\pi)|}{|\lambda(H_1)|^k} < \infty. \quad (28)$$

We say that $Z_1(s)$ and $Z_2(s)$ are of *finite rank*.

By the Weierstrass Factorisation Theorem ([6], VII.5.14) and (28), there exist entire functions $g_1(s)$, $g_2(s)$ such that

$$Z_i(s) = s^{n_i} e^{g_i(s)} \prod_{\rho \in R_i \setminus \{0\}} \left(1 - \frac{s}{\rho} \right) \exp \left(\frac{s}{\rho} + \frac{s^2}{2\rho^2} \cdots + \frac{s^k}{k\rho^k} \right), \quad (29)$$

where n_i is the order of the zero of $Z_i(s)$ at $s = 0$.

From (24) we recall the equation

$$\left(\frac{d}{ds} \right)^j \log Z_{P,\sigma}(s) = \sum_{\pi \in \hat{G}(\Gamma)} N_\Gamma(\pi) \left(\frac{d}{ds} \right)^{j-1} \sum_{\lambda \in \Lambda(\pi) \cup \{0\}} \frac{m_\lambda(\pi)}{s - \lambda(H_1)}, \quad (30)$$

where $j \in \mathbb{N}$ is sufficiently large. Let $J = \max(j, k)$. It follows from (29) and (30) that

$$\left(\frac{d}{ds} \right)^{J-1} (g'_1(s) - g'_2(s)) + \sum_{\rho \in R_1} \frac{(-1)^{J-1}(J-1)!}{(s-\rho)^J} - \sum_{\rho \in R_2} \frac{(-1)^{J-1}(J-1)!}{(s-\rho)^J}$$

is equal to

$$\sum_{\pi \in \hat{G}(\Gamma)} \sum_{\lambda \in \Lambda(\pi) \cup \{0\}} N_\Gamma(\pi) m_\lambda(\pi) \frac{(-1)^{J-1}(J-1)!}{(s - \lambda(H_1))^J}.$$

Remembering that the zeros of $Z_i(s)$ are included with multiplicity in $R_i(s)$ and bearing in mind the expression (19) for the vanishing order of $Z_{P,\sigma}(s)$, we see that this implies that

$$\left(\frac{d}{ds} \right)^{J-1} (g'_1(s) - g'_2(s)) = 0.$$

It follows that $g_1(s) - g_2(s)$ is a polynomial of degree at most J . Without loss of generality we may take $g_2(s)$ to be zero, so that $g_1(s)$ is itself a polynomial of degree at most J . Finally, Theorem XI.2.6 of [6] tells us that since $Z_1(s)$ and $Z_2(s)$ are of finite rank and $g_1(s)$ and $g_2(s)$ are both polynomials of degree at most J , it follows that $Z_1(s)$ and $Z_2(s)$ are both of order at most J . \square

Before we give the next lemma we make a couple of definitions. Let G' be a linear connected reductive group with maximal compact subgroup K' and let ξ be a representation of G' . Let \mathfrak{g}' be the complexified Lie algebra of G' , let $\mathfrak{h}' \subset \mathfrak{g}'$ be the diagonal subalgebra and let $\rho' \in (\mathfrak{h}')^*$ be the half sum of the positive roots of the system $(\mathfrak{g}', \mathfrak{h}')$. We say that ξ is *tempered* if for all K' -finite vectors $u, v \in \xi_{K'}$ there exist constants $c_{u,v}$ such that for all $g \in G'$

$$|\langle \xi(g)u, v \rangle| \leq c_{u,v} \int_{K'} e^{-\rho'(H(g^{-1}k))} dk,$$

where $H(g^{-1}k)$ denotes the logarithm of the split part of $g^{-1}k$ under the Iwasawa decomposition and $\langle \cdot, \cdot \rangle$ is the inner product on V_ξ . An admissible representation of G' is called *standard* if it is induced from an irreducible tempered representation of $M' \subset G'$, where $P' = M'A'N'$ is a parabolic subgroup of G' .

The Weyl group $W(G, A)$ has two elements, let w be the nontrivial element therein. Then w acts on M by conjugation and for a representation σ of M we can define the representation ${}^w\sigma$ by ${}^w\sigma(m) = \sigma(wmw^{-1})$.

Lemma 2.7 *Suppose that $\sigma \cong {}^w\sigma$ as K_M -modules. Then the functions $Z_{P,\sigma}(s)$ and $Z_{P,\sigma}(1-s)$ have the same poles and zeros with multiplicity.*

Proof: By Theorem 2.4 $\lambda \in \mathfrak{a}^*$, the vanishing order of $Z_{P,\sigma}(s)$ at the point $s = \lambda(H_1)$ is equal to $\sum_{\pi \in \hat{G}} N_\Gamma(\pi) m_\lambda(\pi)$. We shall show that

$$m_\lambda(\pi) = m_{-\lambda-2\rho_P}(\pi) \quad (31)$$

for all $\pi \in \hat{G}$. Since $-2\rho_P(H_1) = 1$ the lemma will follow.

Let $\pi \in \hat{G}$. Recall from (18) that

$$m_\lambda(\pi) = \sum_{p,q \geq 0} (-1)^{p+q} \dim \left(H^q(\mathfrak{n}, \pi_K)^\lambda \otimes \bigwedge^p \mathfrak{p}_M \otimes V_\sigma \right)^{K_M}.$$

The global character Θ_π^G of π on G can be written as a linear combination with integer coefficients of characters of standard representations ([26], Chapter X, §10.2). We are interested in the values taken by Θ_π^G on MA^- . By [26], Proposition 10.19, the only characters which are non-zero on MA are those characters of representations induced from $P = MAN$. So there exist $n \in \mathbb{N}$ and for all $1 \leq i \leq n$ integers c_i , tempered representations ξ_i of M and $\nu_i \in \mathfrak{a}^*$ such that

$$\Theta_\pi^G = \sum_{i=1}^n c_i \Theta_{\pi^i}^G, \quad (32)$$

where $\pi^i = \text{Ind}_P^G(\xi_i \otimes \nu_i)$. From [20], Theorem 3.6 and (17) it follows that for all regular $ma \in MA^-$ we have

$$\Theta_\pi^G(ma) = \Theta_{H^\bullet(\mathfrak{n}, \pi_K)}^{MA}(ma) \frac{\det(a| \bigwedge^4 \mathfrak{n})}{\det(1 - ma| \mathfrak{n})},$$

and the same holds for all π^i . Together with (32) this implies that

$$\Theta_{H^\bullet(\mathfrak{n}, \pi_K)}^{MA}(ma) = \sum_{i=1}^n \Theta_{H^\bullet(\mathfrak{n}, \pi_K^i)}^{MA}(ma)$$

for all regular $ma \in MA^-$. Substituting this into (22) and proceeding as in (23) we see that it suffices to show (31) for the representations π^i .

The Weyl group element w acts on the group MA by conjugation, which has the effect of swapping the two components. For $\nu \in \mathfrak{a}^*$ we therefore have $w\nu = -\nu$. Recall that for a representation ξ of M we let ${}^w\xi$ denote the representation ${}^w\xi(m) = \xi(wmw^{-1})$. By [26], Theorem 8.38, the representations π^i and ${}^w\pi^i = \text{Ind}_P^G({}^w\xi_i \otimes (-\nu_i))$ are equivalent. From the definition of the induced group action we can see that

$$H^q(\mathfrak{n}, \pi_K^i)^\lambda = H^q(\mathfrak{n}, {}^w\pi_K^i)^{-\lambda-2\rho_P}.$$

Thus we see that

$$\dim \left(H^q(\mathfrak{n}, \pi_K^i)^\lambda \otimes \bigwedge^p \mathfrak{p}_M \otimes V_\sigma \right)^{K_M}$$

is equal to

$$\dim \left(H^q(\mathfrak{n}, {}^w\pi_K^i)^{-\lambda-2\rho_P} \otimes \bigwedge^p \mathfrak{p}_M \otimes V_\sigma \right)^{K_M}.$$

Using the notation of Proposition 1.11 we note that the K_M -types satisfy ${}^w\text{triv}_{K_M} = \text{triv}_{K_M}$, ${}^w\det_{K_M} = \det_{K_M}$ and ${}^w\delta_{l,m} = \delta_{m,l} \cong \delta_{-m,-l}$. Lemma 1.15 and the isomorphism of K_M -modules

$$\mathfrak{n} \cong \delta_{2,2} \oplus \delta_{2,-2}$$

tell us that the K_M -modules $\bigwedge^p \mathfrak{p}_M$ and \mathfrak{n} are invariant under the action of w and we have assumed that ${}^w\sigma \cong \sigma$ as K_M -modules. Since w^2 is the identity element of $W(G, A)$ we can conclude that

$$\dim \left(H^q(\mathfrak{n}, \pi_K^i)^\lambda \otimes \bigwedge^p \mathfrak{p}_M \otimes V_\sigma \right)^{K_M}$$

is equal to

$$\dim \left(H^q(\mathfrak{n}, \pi_K^i)^{-\lambda-2\rho_P} \otimes \bigwedge^p \mathfrak{p}_M \otimes V_\sigma \right)^{K_M}.$$

Hence $m_\lambda(\pi^i) = m_{-\lambda-2\rho_P}(\pi^i)$ and the lemma follows. \square

Theorem 2.8 Suppose that $\sigma \cong {}^w\sigma$ as K_M -modules. Then there exists a polynomial $G(s)$ such that $Z_{P,\sigma}(s)$ satisfies the functional equation

$$Z_{P,\sigma}(s) = e^{G(s)} Z_{P,\sigma}(1-s).$$

Proof: Let R_i, n_i be as above for $i = 1, 2$. We saw in the proof of Lemma 2.6 that there exist entire functions $Z_1(s)$ and $Z_2(s)$ of finite order such that

$$Z_{P,\sigma}(s) = \frac{Z_1(s)}{Z_2(s)}$$

and polynomials $g_1(s)$ and $g_2(s)$ such that

$$Z_i(s) = s^{n_i} e^{g_i(s)} \prod_{\rho \in R_i \setminus \{0\}} \left(1 - \frac{s}{\rho}\right) \exp\left(\frac{s}{\rho} + \frac{s^2}{2\rho^2} \cdots + \frac{s^k}{k\rho^k}\right).$$

Lemma 2.7 tells us that $Z_{P,\sigma}(s)$ and $Z_{P,\sigma}(1-s)$ have the same poles and zeros. Hence we can in the same way conclude that there exist entire functions $W_1(s)$ and $W_2(s)$ of finite order such that

$$Z_{P,\sigma}(1-s) = \frac{W_1(s)}{W_2(s)}$$

and polynomials $h_1(s)$ and $h_2(s)$ such that

$$W_i(s) = s^{n_i} e^{h_i(s)} \prod_{\rho \in R_i \setminus \{0\}} \left(1 - \frac{s}{\rho}\right) \exp\left(\frac{s}{\rho} + \frac{s^2}{2\rho^2} \cdots + \frac{s^k}{k\rho^k}\right).$$

Setting

$$G(s) = g_1(s) + h_2(s) - g_2(s) - h_1(s)$$

we can see that the claimed functional equation does indeed hold. \square

2.4 The Ruelle zeta function

For any finite-dimensional virtual representation σ of M we define, for $\text{Re}(s)$ large, the *generalised Ruelle zeta function*

$$R_{\Gamma,\sigma}(s) = \prod_{[\gamma] \in \mathcal{E}_P^p(\Gamma)} \prod_{I \subset R_\gamma} \det(1 - e^{-sn_I l_\gamma} \gamma^{n_I} | V_\sigma)^{\chi_I(\gamma)}.$$

We have the following theorem giving a relationship between the generalised Selberg zeta function and the generalised Ruelle zeta function.

Theorem 2.9 Let σ be a finite dimensional virtual representation of M . The function $R_{\Gamma,\sigma}(s)$ extends to a meromorphic function on \mathbb{C} . More precisely

$$R_{\Gamma,\sigma}(s) = \prod_{q=0}^4 Z_{P,(\wedge^q \mathfrak{n} \otimes \sigma)} \left(s + \frac{q}{4}\right)^{(-1)^q}.$$

Proof: Let $\gamma \in \mathcal{E}_P(\Gamma)$ let $\mu(\gamma) \in \mathbb{N}$ be the least such that $\gamma = \gamma_0^{\mu(\gamma)}$ for some $\gamma_0 \in \mathcal{E}_P^p(\Gamma)$. We compute at first

$$\begin{aligned} \log R_{\Gamma,\sigma}(s) &= \sum_{[\gamma] \in \mathcal{E}_P^p(\Gamma)} \sum_{I \subset R_\gamma} \chi_I(\gamma) \text{tr} \log(1 - e^{-sn_I l_\gamma} \gamma^{n_I} | V_\sigma) \\ &= \sum_{[\gamma] \in \mathcal{E}_P^p(\Gamma)} \sum_{I \subset R_\gamma} \chi_I(\gamma) \sum_{m=1}^{\infty} \frac{1}{m} e^{-sm n_I l_\gamma} \text{tr} \sigma(b_\gamma^{n_I}) \\ &= \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \sum_{I \subset R_\gamma} \chi_I(\gamma) \frac{e^{-sn_I l_\gamma}}{\mu(\gamma)} \text{tr} \sigma(b_\gamma^{n_I}). \end{aligned}$$

Similarly, for $\tau_q = \bigwedge^q \bar{\mathbf{n}} \otimes \sigma$ we have that $\log Z_{P, \tau_q}(s)$ equals

$$\begin{aligned}
& \sum_{[\gamma] \in \mathcal{E}_P^p(\Gamma)} \sum_{I \subset R_\gamma} \chi_I(\gamma) \sum_{n=0}^{\infty} \text{tr} \log \left((1 - e^{-sn_I l_\gamma} \gamma^{n_I} | V_{\tau_q} \otimes S^n(\mathbf{n})) \right) \\
&= \sum_{[\gamma] \in \mathcal{E}_P^p(\Gamma)} \sum_{I \subset R_\gamma} \chi_I(\gamma) \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{-1}{m} e^{-smn_I l_\gamma} \text{tr} \left(\gamma^{mn_I} | V_{\tau_q} \otimes S^n(\mathbf{n}) \right) \\
&= - \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \sum_{I \subset R_\gamma} \chi_I(\gamma) \frac{e^{-sn_I l_\gamma}}{\mu(\gamma)} \text{tr} \sigma(b_\gamma^{n_I}) \frac{\text{tr} (b_\gamma^{n_I} | \bigwedge^q \bar{\mathbf{n}})}{\det (1 - (a_\gamma b_\gamma)^{-n_I} | \bar{\mathbf{n}})} \\
&= - \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \sum_{I \subset R_\gamma} \chi_I(\gamma) \frac{e^{-sn_I l_\gamma}}{\mu(\gamma)} \text{tr} \sigma(b_\gamma^{n_I}) \frac{\text{tr} (b_\gamma^{n_I} | \bigwedge^q \bar{\mathbf{n}})}{\text{tr} ((a_\gamma b_\gamma)^{-n_I} | \bigwedge^* \bar{\mathbf{n}})}.
\end{aligned}$$

Since \mathbf{n} is an M -module defined over the reals we conclude that the trace $\text{tr} ((a_\gamma b_\gamma)^{-1} | \bigwedge^* \bar{\mathbf{n}})$ is a real number. Therefore it equals its complex conjugate, which is $\text{tr} (a_\gamma^{-1} b_\gamma | \bigwedge^* \bar{\mathbf{n}})$. Summing over q we get

$$\log R_{\Gamma, \sigma}(s) = \sum_{q=0}^4 (-1)^q \log Z_{P, \tau_q} \left(s + \frac{q}{4} \right).$$

The theorem follows. \square

We shall be interested in the zeta function $R_{P, \sigma}(s)$ in the case where $\sigma = \text{triv} = 1$ is the trivial representation of M and in the case where σ is the following virtual representation of M . Define

$$\tilde{\sigma}^+ = 15 \bigwedge^0 \mathbf{m} + 6 \bigwedge^2 \mathbf{m} + \bigwedge^4 \mathbf{m}$$

and

$$\tilde{\sigma}^- = 10 \bigwedge^1 \mathbf{m} + 3 \bigwedge^3 \mathbf{m}$$

and let $\tilde{\sigma} = \tilde{\sigma}^+ - \tilde{\sigma}^-$, where M acts on $\bigwedge^n \mathbf{m}$ according to Ad^n . The reason for this choice of $\tilde{\sigma}$ is that it allows us to separate the contribution of the non-regular elements of $\mathcal{E}_P(\Gamma)$ from the regular ones, as made clear in the following lemma.

Lemma 2.10 *Let $\gamma \in \mathcal{E}_P(\Gamma)$ with*

$$b_\gamma = \begin{pmatrix} R(\theta) & \\ & R(\phi) \end{pmatrix},$$

where

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Then

$$\text{tr} \tilde{\sigma}(b_\gamma) = \det (1 - b_\gamma : \mathbf{m}/\mathbf{b}) = 4(1 - \cos 2\theta)(1 - \cos 2\phi),$$

where \mathbf{b} is the complexified Lie algebra of the group B . In particular we have $\text{tr} \tilde{\sigma}(b_\gamma) \geq 0$ for all $\gamma \in \mathcal{E}_P(\Gamma)$, and $\text{tr} \tilde{\sigma}(b_\gamma) = 0$ if and only if γ is non-regular.

Proof: As B -modules, we have the isomorphism $\mathbf{m} \cong \mathbf{b} \oplus (\mathbf{m}/\mathbf{b})$. The group B is abelian and so acts trivially on the 2-dimensional Lie algebra \mathbf{b} . Hence, for $0 \leq n \leq 6$, we have the following isomorphism of B -modules

$$\bigwedge^n \mathbf{m} \cong \bigoplus_{p+q=n} \binom{2}{p} \bigwedge^q \mathbf{m}/\mathbf{b}. \quad (33)$$

We set $a_0 = 1$, $a_1 = -3$, $a_2 = 6$, $a_3 = -10$, $a_4 = 15$ and note that, for $k = 0, \dots, 4$, these satisfy

$$\sum_{m=0}^k a_{k-m} \binom{2}{m} = (-1)^k. \quad (34)$$

The B -module isomorphism

$$\bigwedge^* (\mathbf{m}/\mathbf{b}) \cong \sum_{n=0}^4 a_{4-n} \bigwedge^n \mathbf{m} = \tilde{\sigma}$$

then follows from (33) and (34). We can then see that

$$\mathrm{tr} \, \tilde{\sigma}(b_\gamma) = \mathrm{tr} \left(b_\gamma : \bigwedge^* \mathfrak{m}/\mathfrak{b} \right) = \det (1 - b_\gamma : \mathfrak{m}/\mathfrak{b}).$$

The adjoint action of B on $\mathfrak{m}/\mathfrak{b}$ can easily be computed to give the claimed value. Finally we note that $\mathrm{tr} \, \tilde{\sigma}(b_\gamma) = 0$ if and only if $\theta, \phi \in \mathbb{Z}\pi$ which is equivalent to γ being non-regular. \square

The main result of this section is the following theorem.

Theorem 2.11 *The function $R_{\Gamma,1}(s)$ has a double zero at $s = 1$. The function $R_{\Gamma,\tilde{\sigma}}(s)$ has a zero of order eight at $s = 1$. Apart from that, for $\sigma \in \{1, \tilde{\sigma}\}$, all poles and zeros of $R_{\Gamma,\sigma}(s)$ are contained in the union of the interval $[-1, \frac{3}{4}]$ with the five vertical lines given by $\frac{k}{4} + i\mathbb{R}$ for $k = -2, -1, 0, 1, 2$.*

The theorem will follow from the following proposition, which we prove in the remainder of the section.

Proposition 2.12 *The function $Z_{P,1}(s)$ has a double zero at $s = 1$. The function $Z_{P,\tilde{\sigma}}(s)$ has a zero of order eight at $s = 1$. Apart from that, for $\sigma \in \{1, \tilde{\sigma}\}$, all poles and zeros of $Z_{P,\sigma}(s)$ lie in the half-plane $\{\mathrm{Re}(s) \leq \frac{3}{4}\}$. Further, for τ_q being the representation of M on $\bigwedge^q \bar{\mathfrak{n}}$ ($q \in \{1, \dots, 4\}$) obtained from the adjoint representation and $\sigma \in \{\tau_q, \tau_q \otimes \tilde{\sigma}\}$, all poles and zeros of $Z_{P,\sigma}(s)$ lie in the half-plane $\{\mathrm{Re}(s) \leq 1\}$.*

Let us assume for a moment that the proposition has been proved. The representations σ of M considered in the proposition all satisfy the isomorphism of K_M -modules $\sigma \cong {}^w\sigma$, where w is the non-trivial element of the Weyl group $W(G, A)$. Hence we can apply the functional equation given in Theorem 2.8 to see that the poles and zeros of the functions $Z_{P,\sigma}(s)$ all lie in the region $0 \leq \mathrm{Re}(s) \leq 1$. We can then apply Theorem 2.4 to see that the poles and zeros in fact lie in $[0, 1] \cup (\frac{1}{2} + i\mathbb{R})$. Finally, an application of Theorem 2.9 completes the proof of Theorem 2.11.

The proof of the proposition will take up the rest of the section. We see from (19) that a representation $\pi \in \hat{G}$ makes a contribution to the vanishing order of $Z_{P,\sigma}(s)$ only if $m_\lambda(\pi) \neq 0$ for some $\lambda \in \mathfrak{a}^*$. From (23) it follows that, if $\mathrm{tr} \, \pi(\Phi_{s,\sigma}) = 0$ for some $\pi \in \hat{G}$ and finite dimensional representation σ of M , then $m_\lambda(\pi) = 0$ for all $\lambda \in \mathfrak{a}^*$. Thus, if we can show that $\mathrm{tr} \, \pi(\Phi_{s,\sigma}) = 0$ for some π and σ , then we will know that π makes no contribution to the vanishing order of $Z_{P,\sigma}$.

Since $\rho_P(H_1) = -\frac{1}{2}$, we shall be able to prove Proposition 2.12 with the following steps. Firstly, irrespective of the choice of σ , all eigenvalues λ of \mathfrak{a} on $H^\bullet(\mathfrak{n}, \pi_K)$ satisfy

$$\mathrm{Re}(\lambda) \geq -2\rho_P, \tag{35}$$

with equality only if $\pi = \mathrm{triv}$. Secondly, in the case $\sigma \in \{1, \tilde{\sigma}\}$, if π is the trivial representation on G , then $H^4(\mathfrak{n}, \pi_K)^{-2\rho_P} \neq 0$. If $\sigma = 1$ then this gives a double zero at the point $s = 1$ and if $\sigma = \tilde{\sigma}$ this gives a zero of order eight at $s = 1$. Apart from this, for all $\pi \in \hat{G}$ and all $\lambda \in \mathfrak{a}^*$, we have either $\mathrm{tr} \, \pi(\Phi) = 0$, or that $H^\bullet(\mathfrak{n}, \pi_K)^\lambda \neq 0$ implies

$$\mathrm{Re}(\lambda) \geq -\frac{3}{2}\rho_P. \tag{36}$$

The first step will be proven using the Hochschild-Serre spectral sequence in the following two sections. Then the second step will be proven using a result of Hecht and Schmid which relates the action of \mathfrak{a} on $H^\bullet(\mathfrak{n}, \pi_K)$ to the infinitesimal character of π .

2.5 The Hochschild-Serre spectral sequence

We shall need the following proposition.

Proposition 2.13 *Let $P' = M'A'N'$ be a parabolic subgroup of G and let $\mathfrak{a}', \mathfrak{n}'$ be the complexified Lie algebras of A', N' respectively. Define a partial order $>_{\mathfrak{a}'}$ on the dual space \mathfrak{a}'^* of \mathfrak{a}' by $\mu > \nu$ if and only if $\mu - \nu$ is a non-zero linear combination with positive integral coefficients of positive roots of $(\mathfrak{g}, \mathfrak{a})$.*

Let $\lambda \in \mathfrak{a}'^$ and $0 \leq p < \dim \mathfrak{n}$ be such that $H^p(\mathfrak{n}, \pi_K)^\lambda \neq 0$. Then there exists $\mu \in \mathfrak{a}'^*$ with $\lambda >_{\mathfrak{a}'} \mu$ such that $H^{\dim \mathfrak{n}}(\mathfrak{n}, \pi_K)^\mu \neq 0$.*

Proof: This follows from [20], Proposition 2.32 and the isomorphism (17). \square

Let $M_0 \subset G$ be the subgroup of diagonal matrices with each diagonal entry equal to ± 1 , let $A_0 \subset G$ be the subgroup of diagonal matrices with positive entries and let $N_0 \subset G$ be the subgroup of upper triangular matrices with ones on the diagonal. Then $P_0 = M_0 A_0 N_0$ is the minimal parabolic subgroup of G . Let \mathfrak{a}_0 and \mathfrak{n}_0 be the complexified Lie algebras of A_0 and N_0 respectively, and let \mathfrak{a}_0^* be the dual of \mathfrak{a}_0 . Let $\rho_0 \in \mathfrak{a}_0^*$ be defined as follows:

$$\rho_0 \left(\begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix} \right) = 3a + 2b + c.$$

Then ρ_0 is the half-sum of the positive roots of the system $(\mathfrak{g}, \mathfrak{a}_0)$. Let $\mathfrak{a}_{0,\mathbb{R}}$ be the real Lie algebra of A_0 , which we may consider as a subalgebra of \mathfrak{a}_0 . Let $\mathfrak{a}_{0,\mathbb{R}}^-$ be the set of all $X \in \mathfrak{a}_{0,\mathbb{R}}$ such that $\alpha(X) < 0$ for all positive roots α of the system $(\mathfrak{g}, \mathfrak{a}_0)$. Let $\mathfrak{a}_{0,\mathbb{R}}^{*,+}$ be the set of all $\nu \in \mathfrak{a}_{0,\mathbb{R}}^*$ such that $\nu(X) < 0$ for all $X \in \mathfrak{a}_{0,\mathbb{R}}^-$. Then for all $\nu \in \mathfrak{a}_{0,\mathbb{R}}^{*,+}$ we have $\nu = \sum_{\alpha} \lambda_{\alpha} \alpha$, where the sum is over the positive roots of $(\mathfrak{g}, \mathfrak{a}_0)$ and every $\lambda_{\alpha} > 0$.

For $\pi \in \hat{G}$ a *matrix-coefficient* of π is any function $G \rightarrow \mathbb{C}$ of the form

$$f_{u,v}(g) = \langle \pi(g)u, v \rangle$$

for some $u, v \in \pi$.

Lemma 2.14 *Let $\pi \in \hat{G}$ and let $\lambda \in \mathfrak{a}_0^*$ be such that $H^{\bullet}(\mathfrak{n}_0, \pi_K)^{\lambda} \neq 0$. Then $\lambda \in -2\rho_0 + \mathfrak{a}_{0,\mathbb{R}}^{*,+}$, except in the case $\pi = \text{triv}$, when $H^6(\mathfrak{n}_0, \pi_K)^{-2\rho_0} \neq 0$ and other than this $H^{\bullet}(\mathfrak{n}_0, \text{triv})^{\lambda} \neq 0$ implies $\lambda \in -2\rho_0 + \mathfrak{a}_{0,\mathbb{R}}^{*,+}$.*

Proof: According to [20], Theorem 4.16, there exists a countable set $\mathcal{E}(\pi) \subset \mathfrak{a}_1^*$ and a collection of polynomial functions $p_{u,v}^{\nu}$ indexed by $u, v \in \pi_K$ and $\nu \in \mathcal{E}(\pi)$ such that if $f_{u,v}$, for $u, v \in \pi_K$, is a matrix coefficient of π then:

$$f_{u,v}(\exp X) = \sum_{\nu \in \mathcal{E}(\pi)} p_{u,v}^{\nu}(X) e^{(\nu + \rho_0)(X)}, \quad (37)$$

for all $X \in \mathfrak{a}_{0,\mathbb{R}}^-$. We assume that $\mathcal{E}(\pi)$ is minimal and hence that each polynomial $p_{u,v}^{\nu}$ is non-zero.

If $\pi = \text{triv}$ then every matrix coefficient of π is a constant function so in (37) we have $\mathcal{E}(\pi) = \{-\rho_0\}$ and $p_{u,v}^{-\rho_0} = C_{u,v}$, where $C_{u,v}$ is a constant for all $u, v \in \pi$.

If $\pi \in \hat{G} \setminus \{\text{triv}\}$ then π is infinite dimensional so by [25], Theorem 5.1 all the matrix coefficients of π vanish at infinity. Hence, for all $\nu \in \mathcal{E}(\pi)$ and for all $X \in \mathfrak{a}_{0,\mathbb{R}}^-$, we have $(\nu + \rho_0)(X) < 0$. In other words, for all $\nu \in \mathcal{E}(\pi)$ we have $\nu \in -\rho_0 + \mathfrak{a}_{0,\mathbb{R}}^{*,+}$.

We have assumed that $H^{\bullet}(\mathfrak{n}_0, \pi_K)^{\lambda} \neq 0$. Let $>_{\mathfrak{a}_0}$ be the partial order defined on \mathfrak{a}_0^* by $\mu >_{\mathfrak{a}_0} \nu$ if and only if $\mu - \nu$ is a linear combination with positive integral coefficients of positive roots of $(\mathfrak{g}, \mathfrak{a}_0)$. By Proposition 2.13 there exists $\mu \in \mathfrak{a}_0^*$ such that $H^6(\mathfrak{n}_0, \pi_K)^{\mu} \neq 0$ and $\lambda \geq_{\mathfrak{a}_0} \mu$.

Let $\Lambda = \{\nu + \rho_0 \in \mathfrak{a}_0^* : H^6(\mathfrak{n}_0, \pi_K)^{\nu} \neq 0\}$ and let Λ^{\min} be the set of elements of Λ which are minimal with respect to $>_{\mathfrak{a}_0}$. Let $\mathcal{E}^{\min}(\pi)$ be the set of $\nu \in \mathcal{E}(\pi)$ which are minimal with respect to $>_{\mathfrak{a}_0}$. Theorem 4.25 of [20] says that $\Lambda^{\min} = \mathcal{E}(\pi)^{\min}$.

If $\pi = \text{triv}$ then $\Lambda^{\min} = \{-\rho_0\}$ and the claims of the proposition follow from the definition of Λ and [20], Proposition 2.32 as quoted above.

If $\pi \in \hat{G} \setminus \{\text{triv}\}$ then $\Lambda^{\min} = \mathcal{E}(\pi)^{\min}$ implies that there exists $\nu \in \mathcal{E}(\pi)$ such that $\lambda \geq_{\mathfrak{a}_0} \nu - \rho_0$. We saw above that $\nu \in -\rho_0 + \mathfrak{a}_{0,\mathbb{R}}^{*,+}$ so the claim follows. \square

Proposition 2.15 *Let $\pi \in \hat{G}$ and let $\lambda \in \mathfrak{a}^*$ be such that $H^{\bullet}(\mathfrak{n}, \pi_K)^{\lambda} \neq 0$. Then $\text{Re}(\lambda) \geq -2\rho_P$, with equality only if $\pi = \text{triv}$.*

Proof: Let $\mathfrak{n}_M = \mathfrak{n}_0 \cap \mathfrak{m}$ and $\mathfrak{a}_M = \mathfrak{a}_0 \cap \mathfrak{m}$. Then $\mathfrak{n}_0 = \mathfrak{n}_M \oplus \mathfrak{n}$ and $\mathfrak{a}_0 = \mathfrak{a} \oplus \mathfrak{a}_M$. In [23, 22] a filtered complex is constructed so that the spectral sequence derived from it has

$$E_2^{p,q} \cong H^p(\mathfrak{n}_M, H^q(\mathfrak{n}, \pi_K))$$

and

$$E_{\infty}^{p,q} \cong \text{Gr}^p H^{p+q}(\mathfrak{n}_0, \pi_K),$$

where $H^{p+q}(\mathfrak{n}_0, \pi_K)$ is appropriately filtered. This is the so-called *Hochschild-Serre spectral sequence*.

By Proposition 2.13 it suffices to prove the proposition in the case

$$H^4(\mathfrak{n}, \pi_K)^{\lambda} \neq 0.$$

In this case it then follows that

$$H^2(\mathfrak{n}_M, H^4(\mathfrak{n}, \pi_K)^{\lambda}) \neq 0$$

and so there exists $\lambda_M \in \mathfrak{a}_M^*$ such that

$$H^2(\mathfrak{n}_M, H^4(\mathfrak{n}, \pi_K)^{\lambda})^{\lambda_M} \neq 0.$$

Since A acts trivially on \mathfrak{n}_M , this equals

$$H^2(\mathfrak{n}_M, H^4(\mathfrak{n}, \pi_K))^{\lambda+\lambda_M} = (E_2^{2,4})^{\lambda+\lambda_M},$$

where we consider $\lambda+\lambda_M$ as an element of $\mathfrak{a}_0^* = \mathfrak{a}^* \oplus \mathfrak{a}_M^*$. For all r we have that $E_r^{p,q} \neq 0$ only if $0 \leq p \leq 2$ and $0 \leq q \leq 4$, so it follows that $E_2^{2,4} = E_{\infty}^{2,4}$. Since the action of A_0 commutes with the differentials of the spectral sequence it follows that

$$(E_{\infty}^{2,4})^{\lambda+\lambda_M} \neq 0$$

and hence that

$$H^6(\mathfrak{n}_0, \pi_K)^{\lambda+\lambda_M} \neq 0.$$

The proposition then follows from Lemma 2.14 by projection of $\lambda + \lambda_M$ onto the \mathfrak{a}^* component. \square

2.6 Contribution of the trivial representation

For a K_M -module η let 2η denote the module $\eta \oplus \eta$. We shall need the following:

Lemma 2.16 *We have the following isomorphisms of K_M -modules:*

$$\begin{aligned} \bigwedge^0 \mathfrak{p}_M &\cong \text{triv} \\ \bigwedge^1 \mathfrak{p}_M &\cong \delta_{2,0} \oplus \delta_{0,2} \\ \bigwedge^2 \mathfrak{p}_M &\cong 2\delta \oplus \delta_{2,2} \oplus \delta_{2,-2} \\ \bigwedge^3 \mathfrak{p}_M &\cong \delta_{2,0} \oplus \delta_{0,2} \\ \bigwedge^4 \mathfrak{p}_M &\cong \text{triv}. \end{aligned}$$

and

$$\begin{aligned} \bigwedge^0 \mathfrak{m} &\cong \text{triv} \\ \bigwedge^1 \mathfrak{m} &\cong 2\delta \oplus \delta_{2,0} \oplus \delta_{0,2} \\ \bigwedge^2 \mathfrak{m} &\cong \text{triv} \oplus 2\delta \oplus 2(\delta_{2,0} \oplus \delta_{0,2}) \oplus \delta_{2,2} \oplus \delta_{2,-2} \\ \bigwedge^3 \mathfrak{m} &\cong 2\text{triv} \oplus 2\text{triv} \oplus 2(\delta_{2,0} \oplus \delta_{0,2} \oplus \delta_{2,2} \oplus \delta_{2,-2}) \\ \bigwedge^4 \mathfrak{m} &\cong \text{triv} \oplus 2\delta \oplus 2(\delta_{2,0} \oplus \delta_{0,2}) \oplus \delta_{2,2} \oplus \delta_{2,-2} \end{aligned}$$

Proof: The isomorphisms for \mathfrak{p}_M were given in Lemma 1.15. For \mathfrak{m} note that K_M acts on \mathfrak{m} by the adjoint representation and we can compute

$$\mathfrak{m} \cong 2\delta \oplus \delta_{2,0} \oplus \delta_{0,2}.$$

The other isomorphisms follow straightforwardly from this. \square

Proposition 2.17 *Let π be the trivial representation on G .*

For σ the trivial representation on M , the representation π contributes a double zero of $Z_{P,\sigma}(s)$ at the point $s = 1$. For $\sigma = \tilde{\sigma}$ the representation π contributes a zero of $Z_{P,\sigma}(s)$ of order eight at the point $s = 1$. In both cases, all other poles and zeros contributed by π are in $\{\operatorname{Re}(s) \leq \frac{3}{4}\}$.

Proof: The space $H_0(\mathfrak{n}, \pi_K) = \pi_K / \mathfrak{n}\pi_K$ is one dimensional with trivial \mathfrak{a} -action. The action of \mathfrak{a} on \mathfrak{n} is given by $\frac{1}{2}\rho_P$, so the isomorphism (17) tells us that \mathfrak{a} acts on the one dimensional space $H^4(\mathfrak{n}, \pi_K)$ according to $-2\rho_P$. Proposition 2.13 tells us that for $q = 0, 1, 2, 3$ and $\lambda \in \mathfrak{a}^*$, $H^q(\mathfrak{n}, \pi_K)^\lambda \neq 0$ implies $\lambda \geq -\frac{3}{2}\rho_P$. Since $-2\rho_P(H_1) = 1$ this gives a pole or zero at $s = 1$ and evaluation of the relevant characters at H_1 also gives the other poles and zeros contributed by π in $\{\operatorname{Re}(s) \leq \frac{3}{4}\}$.

It remains to compute the vanishing order of $Z_{P,\sigma}(s)$ for $\sigma \in \{1, \tilde{\sigma}\}$ at the point $s = 1$. Since $\dim H^4(\mathfrak{n}, \pi_K)^{-2\rho_P} = 1$, we get from (19) the following expression for the vanishing order:

$$N_\Gamma(\pi) \sum_{p \geq 0} (-1)^p \dim \left(\bigwedge^p \mathfrak{p}_M \otimes V_\sigma \right)^{K_M}.$$

For $\sigma = 1$ this is equal to

$$N_\Gamma(\pi) \sum_{p \geq 0} (-1)^p \dim \left(\bigwedge^p \mathfrak{p}_M \right)^{K_M}$$

and for $\sigma = \tilde{\sigma}$ to

$$N_\Gamma(\pi) \sum_{p, q \geq 0} (-1)^{p+q} \dim \left(\bigwedge^p \mathfrak{p}_M \otimes a_q \bigwedge^q \mathfrak{m} \right)^{K_M},$$

where $a_0 = 15$, $a_1 = 10$, $a_2 = 6$, $a_3 = 3$, $a_4 = 1$ and $a_q = 0$ for $q \geq 5$. The only functions on $L^2(\Gamma \backslash G)$ invariant under the action of G are the constant functions, hence $N_\Gamma(\pi) = 1$. We can then use Lemma 2.16 to see that the above expressions take the claimed values. \square

2.7 Contribution of the other representations

The following proposition gives a relationship between the action of \mathfrak{a} on $H^\bullet(\mathfrak{n}, \pi_K)$ and the infinitesimal character of π .

Proposition 2.18 *Let $\pi \in \hat{G}$.*

Suppose $H^\bullet(\mathfrak{n}, \pi_K)^\lambda \neq 0$ for some $\lambda \in \mathfrak{a}^$. Then $\lambda = w\Lambda_\pi|_{\mathfrak{a}} - \rho_P$, where $w \in W(\mathfrak{g}, \mathfrak{h})$ and Λ_π is a representative of the infinitesimal character of π . Moreover, for $p \in \mathbb{Z}$ we have*

$$H^p(\mathfrak{n}, \pi_K) = \bigoplus_{w \in W(\mathfrak{g}, \mathfrak{h})} H^p(\mathfrak{n}, \pi_K)^{w\Lambda_\pi - \rho_P}.$$

Proof: This follows from [20], Corollary 3.32 and the isomorphism (17). \square

In light of this proposition, in order to complete the proof of Proposition 2.12, and hence of Theorem 2.11, it will be sufficient to show that for all $\pi \in \hat{G} \setminus \{\text{triv}\}$ and for $\sigma \in \{1, \tilde{\sigma}\}$ either $\operatorname{tr} \pi(\Phi_\sigma) = 0$ or the infinitesimal character Λ_π of π satisfies

$$\operatorname{Re}(w\Lambda_\pi)|_{\mathfrak{a}} \geq -\frac{1}{2}\rho_P \quad \text{or} \quad -\rho_P \geq \operatorname{Re}(w\Lambda_\pi)|_{\mathfrak{a}}$$

for all $w \in W(\mathfrak{g}, \mathfrak{h})$.

We first consider the elements of \hat{G} induced from parabolic subgroups other than $P = MAN$.

We say that two parabolics $P' = M'A'N'$ and $P'' = M''A''N''$, where

$$A', A'' \subset A_0 = \{\operatorname{diag}(a, b, c, (abc)^{-1}) | a, b, c > 0\}$$

are *associate* if there exists a member w of the Weyl group $W(\mathfrak{g}, \mathfrak{h})$ such that $w^{-1}MAw = M'A'$. Representations of G induced from associate parabolics are equivalent.

Up to association G has four parabolic subgroups: P defined above; the minimal parabolic $P_0 = M_0 A_0 N_0$ where $M_0 \cong \{\pm 1\}$ and N_0 is the group of real, upper triangular matrices with ones on the diagonal; the parabolic P' with Langlands decomposition $P' = M' A' N'$ where $M' \cong \mathrm{SL}_2^\pm(\mathbb{R}) \times \{\pm 1\}$, $A' = \{\mathrm{diag}(a, a, b, a^{-2}b^{-1}) | a, b > 0\}$ and N' is the group of real, upper triangular matrices with ones on the diagonal and otherwise with zeros in the second column; and the parabolic P'' with Langlands decomposition $P'' = M'' A'' N''$ where $M'' \cong \mathrm{SL}_3^\pm(\mathbb{R})$, $A'' = \{\mathrm{diag}(a, a, a, a^{-3}) | a > 0\}$ and N'' is the group of real, upper triangular matrices with ones on the diagonal whose only non-zero entries above the diagonal are in the rightmost column.

Proposition 2.19 *Let π be a principal series or complementary series representation induced from the parabolic $\bar{P} = \bar{M}\bar{A}\bar{N}$, where $\bar{P} = P_0, P'$ or P'' and let σ be a finite dimensional representation on M . Then $\mathrm{tr} \pi(\Phi_\sigma) = 0$ so π contributes no zeros or poles to $Z_{P,\sigma}(s)$.*

Proof: Let Θ_π be the global character of the irreducible unitary representation π , so that

$$\mathrm{tr} \pi(\Phi_\sigma) = \int_G \Phi_\sigma(x) \Theta_\pi(x) dx.$$

The Weyl integration formula can be applied (see [8], p908) to give us

$$\mathrm{tr} \pi(\Phi_\sigma) = \sum_L \frac{1}{|W(L)|} \int_{A^+ L} \int_{M/L} f_\sigma(mlm^{-1}) dm \Theta_\pi(al) d(al) dl,$$

where the sum is over conjugacy classes of Cartan subgroups L of M , and we denote by $W(L) = W(L, M)$ the Weyl group of L in M , by f_σ an Euler-Poincaré function for σ and $d(al)$ is an explicitly given function on $A^+ L$. Proposition 1.4 of [8] tells us that for $L \neq B$

$$\int_{M/L} f_\sigma(mlm^{-1}) dm = 0,$$

hence

$$\mathrm{tr} \pi(\Phi_\sigma) = \frac{1}{|W(B)|} \int_{A^+ B} \int_{M/B} f_\sigma(mbm^{-1}) dm \Theta_\pi(ab) d(ab) db.$$

The character Θ_π of π is non-zero only on Cartan subgroups of G that are G -conjugate to Cartan subgroups of $\bar{M}\bar{A}$ (see [26], Proposition 10.19). The subgroup BA is not G -conjugate to any Cartan subgroup of $\bar{M}\bar{A}$, so it follows that $\mathrm{tr} \pi(\Phi_\sigma) = 0$. \square

Now let $\pi = \mathrm{Ind}_P^G(\xi \otimes \nu)$ for some $\xi \in \hat{M}$ and $\nu \in \mathfrak{a}^*$. For $\tau \in \widehat{K_M}$ let $P_\tau : V_\xi \rightarrow V_\xi(\tau)$ be the projection onto the τ -isotype. For any function f on G which is sufficiently smooth and of sufficient decay the operator $\pi(f)$ is of trace class. Its trace is

$$\sum_{\tau \in \widehat{K_M}} \int_K \int_{MAN} a^{\nu + \rho_P} f(k^{-1}mank) \mathrm{tr} P_\tau \xi(m) P_\tau dman dk.$$

Plugging in the test function $f = \Phi_\sigma$, where $\sigma \in \{triv, \tilde{\sigma}\}$, this gives us, as in [9]:

$$\mathrm{tr} \pi(\Phi_\sigma) = \int_{A^+} C(a) \mathrm{tr} \xi(f_\sigma) da,$$

where $C(a)$ depends only on a and f_σ is an Euler-Poincaré function on M attached to the representation σ . We can see that $\mathrm{tr} \pi(\Phi_\sigma)$ is non-zero only if $\mathrm{tr} \xi(f_\sigma)$ is also.

Theorem 2.11 now follows from Proposition 1.12.

3 A Prime Geodesic Theorem for $\mathrm{SL}_4(\mathbb{R})$

We continue using the notation defined in the previous sections, in particular we take $G = \mathrm{SL}_4(\mathbb{R})$ and take $\Gamma \subset G$ to be a discrete, cocompact subgroup. For $\gamma \in \Gamma$ let $N(\gamma) = e^{t\gamma}$ and define for $x > 0$

$$\pi(x) = \sum_{\substack{[\gamma] \in \mathcal{E}_P^B(\Gamma) \\ N(\gamma) \leq x}} \chi_1(\Gamma_\gamma), \quad \text{and} \quad \tilde{\pi}(x) = \sum_{\substack{[\gamma] \in \mathcal{E}_P^{B, \mathrm{reg}}(\Gamma) \\ N(\gamma) \leq x}} \chi_1(\Gamma_\gamma) \mathrm{tr} \tilde{\sigma}(b_\gamma),$$

where Γ_γ is the centraliser of γ in Γ , we denote by $\chi_1(\Gamma_\gamma)$ the first higher Euler characteristic and $\tilde{\sigma}$ is the virtual representation defined in Section 2.4. We recall from Proposition 1.10 that the first higher Euler characteristics are all positive and from Lemma 2.10 that the traces $\text{tr } \tilde{\sigma}(b_\gamma)$ are also all positive, so both $\pi(x)$ and $\tilde{\pi}(x)$ are monotonically increasing functions.

Theorem 3.1 (Prime Geodesic Theorem) *For $x \rightarrow \infty$ we have*

$$\pi(x) \sim \frac{2x}{\log x} \quad \text{and} \quad \tilde{\pi}(x) \sim \frac{8x}{\log x}.$$

More sharply,

$$\pi(x) = 2 \text{li}(x) + O\left(\frac{x^{3/4}}{\log x}\right) \quad \text{and} \quad \tilde{\pi}(x) = 8 \text{li}(x) + O\left(\frac{x^{3/4}}{\log x}\right)$$

as $x \rightarrow \infty$, where $\text{li}(x) = \int_2^x \frac{1}{\log t} dt$ is the integral logarithm.

We also prove the following Prime Geodesic Theorem, which will be needed for our application to class numbers.

Let B^0 be a closed subset of the compact group B with the following properties: it is of measure zero; it is invariant under the map $b \mapsto b^{-1}$ and contains all fixed points of this map; and its complement $B^1 = B \setminus B^0$ in B is homeomorphic to an open subset of Euclidean space each of whose connected components is contractible. The assumption that B^0 contains all fixed points of the map $b \mapsto b^{-1}$ is equivalent to the assumption that B^0 contains all non-regular elements of B . Let $\mathcal{E}_P^{p,1}(\Gamma)$ be the subset of all $[\gamma] \in \mathcal{E}_P^p(\Gamma)$ such that γ is conjugate in G to an element of $A^- B^1$. We define for $x > 0$

$$\pi^1(x) = \sum_{\substack{[\gamma] \in \mathcal{E}_P^{p,1}(\Gamma) \\ N(\gamma) \leq x}} \chi_1(\Gamma_\gamma).$$

Then as in the case of $\pi(x)$ above, $\pi^1(x)$ is a monotonically increasing function.

Theorem 3.2 *For $x \rightarrow \infty$ we have*

$$\pi^1(x) \sim \frac{2x}{\log x}.$$

The proof of these two theorems will occupy the rest of the section.

3.1 Analytic properties of $R_{\Gamma,\sigma}(s)$

This subsection and the following proceed according to the methods of [34] and [35], in which the Selberg zeta function for quotients of the hyperbolic plane are considered. In this section the analogs of a series of lemmas are proved in our context. The next section translates the main theorem of [34] into our context.

Recall from Theorem 2.8 that for a finite dimensional virtual representation σ of M we have the functional equation

$$Z_{P,\sigma}(1-s) = e^{-G(s)} Z_{P,\sigma}(s), \tag{38}$$

where $G(s)$ is a polynomial. Let D be the degree of the polynomial $G(x)$.

Lemma 3.3 *Let H be a half-plane of the form $\{\text{Re}(s) < -(1+\varepsilon)\}$ for some $\varepsilon > 0$ and let σ be a finite dimensional virtual representation of M . Then there exists a constant $C > 0$ such that for $s \in H$ we have*

$$|R'_{\Gamma,\sigma}(s)/R_{\Gamma,\sigma}(s)| \leq C|s|^{D-1}.$$

Proof: From Theorem 2.9 we get the identity

$$R_{\Gamma,\sigma}(s) = \prod_{q=0}^4 Z_{P,(\wedge^q \bar{\mathfrak{n}} \otimes V_\sigma)} \left(s + \frac{q}{4}\right)^{(-1)^q},$$

which implies

$$\frac{R'_{\Gamma,\sigma}(s)}{R_{\Gamma,\sigma}(s)} = \sum_{q=0}^4 (-1)^q \frac{Z'_{P,(\wedge^q \bar{\mathfrak{n}} \otimes V_\sigma)} \left(s + \frac{q}{4}\right)}{Z_{P,(\wedge^q \bar{\mathfrak{n}} \otimes V_\sigma)} \left(s + \frac{q}{4}\right)}. \tag{39}$$

Considering this identity, it will suffice to prove that when K is a half-plane of the form $\{\operatorname{Re}(s) < -\varepsilon\}$ for some $\varepsilon > 0$, there exists a constant $C > 0$ such that for $s \in K$ we have

$$|Z'_{P,(\wedge^q \mathfrak{n} \otimes V_\sigma)}(s)/Z_{P,(\wedge^q \mathfrak{n} \otimes V_\sigma)}(s)| \leq C|s|^{D-1}$$

for all $q = 0, \dots, 4$. Since the proof does not depend on the value of q or on σ we shall abbreviate our notation for the zeta function to $Z_P(s)$, which notation we shall use for the rest of the section.

It follows that

$$-\frac{Z'_P(1-s)}{Z_P(1-s)} = \frac{Z'_P(s)}{Z_P(s)} - G'(s).$$

From the definition (16) of $Z_P(s)$ and Proposition 2.12, we can see that $Z_P(s)$ is both bounded above and bounded away from zero on the half plane $K' = \{\operatorname{Re}(s) > 1 + \varepsilon\}$. It follows that $Z'_P(s)/Z_P(s)$ is bounded on K' . This implies the lemma. \square

For $t > 0$, let $N(t)$, denote the number of poles and zeros of $Z_P(s)$ at points $s = \frac{1}{2} + x$, where $0 < x < t$.

Lemma 3.4 $N(t) = O(t^D)$

Proof: Define $\xi(s) = (Z_P(s))^2 e^{-G(s)}$, where $G(s)$ is the polynomial in the functional equation (38) for $Z_P(s)$. We note that, in light of its role in the functional equation, the polynomial $G(s)$ must satisfy $G(s) = -G(1-s)$. It then follows that

$$\xi(1-s) = \overline{\xi(s)}.$$

Note that $\xi(s)$ is real on the real axis and so $\xi(\bar{s}) = \overline{\xi(s)}$.

Fix a real number $1 < a < 5/4$ and let $t > 0$. Let R be the rectangle defined by the inequalities $1-a \leq \operatorname{Re} s \leq a$ and $-t \leq \operatorname{Im} s \leq t$. We assume that t has been chosen so that no zero or pole occurs on the boundary of R . Then

$$N(t) = \frac{1}{4} \cdot \frac{1}{2\pi i} \int_{\partial R} \frac{\xi'(s)}{\xi(s)} ds - N_0 = \frac{1}{4} \cdot \frac{1}{2\pi} \operatorname{Im} \left(\int_{\partial R} \frac{\xi'(s)}{\xi(s)} ds \right) - N_0,$$

where N_0 is the number of poles and zeros of $Z_P(s)$ on the real line. It then follows from the functional equation for $\xi(s)$ and from the fact that $\xi(\bar{s}) = \overline{\xi(s)}$, that we have

$$N(t) = \frac{1}{2\pi} \operatorname{Im} \left(\int_C \frac{\xi'(s)}{\xi(s)} ds \right) - N_0,$$

where C is the portion of ∂R consisting of the vertical segment from a to $a + it$ plus the horizontal segment from $a + it$ to $\frac{1}{2} + it$.

Now the definition of $\xi(s)$ gives us

$$\frac{\xi'(s)}{\xi(s)} = 2 \frac{Z'_P(s)}{Z_P(s)} - G'(s),$$

so that

$$\begin{aligned} \operatorname{Im} \left(\int_C \frac{\xi'(s)}{\xi(s)} ds \right) &= 2 \operatorname{Im} \left(\int_C \frac{Z'_P(s)}{Z_P(s)} ds \right) - \operatorname{Im} \left(\int_C G'(s) ds \right) \\ &= 2 \operatorname{Im} \left(\int_C \frac{Z'_P(s)}{Z_P(s)} ds \right) - \operatorname{Im} \left(G \left(\frac{1}{2} + it \right) + G(a) \right) \\ &= 2 \operatorname{Im} \left(\int_C \frac{Z'_P(s)}{Z_P(s)} ds \right) + O(t^D). \end{aligned}$$

It thus remains to show that

$$S(t) = \operatorname{Im} \left(\int_C \frac{Z'_P(s)}{Z_P(s)} ds \right) = O(t^D).$$

Note that $S(t)$ is the variation of the argument of $Z_P(s)$ along C . We may extend the definition of $S(t)$ to those values of t for which $\frac{1}{2} + it$ is a pole or zero of $Z_P(s)$ by defining it to be $\lim_{\varepsilon \rightarrow 0} \frac{1}{2}(S(t+\varepsilon) + S(t-\varepsilon))$.

From the definition of $Z_P(s)$ we can see that $S(t) = h(t) + O(1)$, where $h(t)$ is the variation of the argument of $Z_P(s)$ along the segment from $a + it$ to $\frac{1}{2} + it$. The value of $h(t)$ is bounded by a multiple

of the number of zeros of $\operatorname{Re}(Z_P(s))$ on this segment, since the point $Z_P(s)$ cannot move between the right and left half-planes without crossing the imaginary axis. On the segment, the real part of $Z_P(s)$ coincides with

$$f_P(w) = \frac{1}{2}(Z_P(w + it) + Z_P(w - it)),$$

where w runs from $\frac{1}{2}$ to a on the real axis. Since we have assumed that $\frac{1}{2} + it$ is not a zero or pole of $Z_P(s)$, the function $f_P(w)$ is holomorphic in a neighbourhood of the closed disc S , centred at a , of radius $a - \frac{1}{2}$. As this disc contains the interval from $\frac{1}{2}$ to a , we may apply Jensen's Formula ([6], XI.1.2) to conclude that

$$\begin{aligned} h(t) &= O\left(\int_{\partial S} \log |f_P(w)| dw\right) \\ &= O\left(\int_{\partial S} \log |Z_P(w + it) + Z_P(w - it)| dw\right) \\ &= O\left(\int_{\partial S} \log |Z_P(w + it)| + \log |Z_P(w - it)| dw\right) \\ &= O\left(\sum_{i=1,2} \int_{\partial S} \log |Z_i(w + it)| dw + \int_{\partial S} \log |Z_i(w - it)| dw\right). \end{aligned}$$

There remains one final step in the proof of the lemma:

$$|Z_i(x + it)| = e^{O(|t|^D)} \quad (40)$$

uniformly in the strip $1 - a \leq x \leq a$ for $i = 1, 2$. We know that the function $Z_P(s)$ is bounded and bounded away from zero on the line $\operatorname{Re} s = a$. Hence, by Lemma 2.6 and the functional equation (38), we can apply the Phragmén Lindelöf Theorem ([6], Theorem VI.4.1) to give (40). \square

Lemma 3.5 *Given $a < b \in \mathbb{R}$, there exists a sequence (y_n) tending to infinity such that*

$$\left| \frac{R'_\Gamma(x + iy_n)}{R_\Gamma(x + iy_n)} \right| = O(y_n^{2D})$$

for $a < x < b$.

Proof: As in Lemma 3.3 it will suffice to prove the result for $Z_P(s)$.

Using the notation of Lemma 2.6 we have that

$$\frac{Z'_P(s)}{Z_P(s)} = \frac{1}{s}(n_1 - n_2) + g'_1(s) - g'_2(s) + \sum_{i=1,2} (-1)^{i-1} \sum_{\rho \in R_i \setminus \{0\}} s^k \rho^{-k} (s - \rho)^{-1}.$$

Let $t_0 > 2$ be fixed and consider the segment of the line $\operatorname{Re} s = \frac{1}{2}$ given by $\frac{1}{2} + it$ for $t_0 - 1 < t \leq t_0 + 1$. Let $N(t)$ be as above, then by Lemma 3.4 we know that $N(t) = O(t^D)$. It follows immediately that the number of roots on the segment is $O(t_0^D)$.

By the Dirichlet principle, there exists a $\frac{1}{2} + iy$ in the segment whose distance from any pole or zero is greater than C/T^D , for some fixed $C > 0$. We conclude that the portion of the sum

$$\sum_{i=1,2} (-1)^{i-1} \sum_{\rho} s^k \rho^{-k} (s - \rho)^{-1}$$

corresponding to poles and zeros in the segment for $s_x = x + iy$ is $O(y^{2D})$, since $|s_x^k \rho^{-k}| = O(1)$ for these ρ , when $a < x < b$.

To deal with the segments $\frac{1}{2} + it$ for $0 < t \leq t_0 - 1$ and $t_0 + 1 < t < \infty$, we proceed as follows. The portions of the sum $\sum_{i=1,2} (-1)^{i-1} \sum_{\rho} s^k \rho^{-k} (s - \rho)^{-1}$ corresponding to the first and second segments respectively, can be written

$$s_x^k \int_0^{t_p-1} \left(\frac{1}{2} + it\right)^{-k} \left(s_x - \frac{1}{2} - it\right)^{-1} dN(t)$$

and

$$s_x^k \int_{t_p+1}^{\infty} \left(\frac{1}{2} + it\right)^{-k} \left(s_x - \frac{1}{2} - it\right)^{-1} dN(t).$$

Recalling that $N(t) = O(t^D)$, we easily conclude that both of these expressions are $O(T^{2D})$. \square

3.2 Estimating $\psi(x)$ and $\tilde{\psi}(x)$

To simplify notation, in what follows we write γ for an element of $\mathcal{E}_P(\Gamma)$ and γ_0 for a primitive element and recall that $\gamma \in \mathcal{E}_P(\Gamma)$ implies that γ is conjugate in Γ to an element $a_\gamma b_\gamma \in A^-B$. Unless otherwise specified, sums involving γ or γ_0 will be taken over conjugacy classes in $\mathcal{E}_P(\Gamma)$ and $\mathcal{E}_P^p(\Gamma)$ respectively. If γ and γ_0 occur in the same formula it is understood that γ_0 will be the primitive element underlying γ . We denote by $\mathcal{E}_P^{\text{reg}}(\Gamma)$ the subset of regular elements in $\mathcal{E}_P(\Gamma)$. For $x > 0$ let

$$\psi(x) = \sum_{\substack{[\gamma] \in \mathcal{E}_P(\Gamma) \\ N(\gamma) \leq x}} \chi_1(\Gamma_\gamma) l_{\gamma_0}$$

and

$$\tilde{\psi}(x) = \sum_{\substack{[\gamma] \in \mathcal{E}_P^{\text{reg}}(\Gamma) \\ N(\gamma) \leq x}} \chi_1(\Gamma_{\gamma_0}) \text{tr } \tilde{\sigma}(b_\gamma) l_{\gamma_0}.$$

Let σ be a finite dimensional virtual representation of M . We have for $\text{Re}(s) > 1$:

$$\frac{R'_{\Gamma, \sigma}(s)}{R_{\Gamma, \sigma}(s)} = \sum_{\gamma} \chi_1(\Gamma_\gamma) \text{tr } \sigma(b_\gamma) l_{\gamma_0} e^{-sl_\gamma}. \quad (41)$$

The following propositions are analogs of Theorem 2 of [34], from which, in the next subsection, we prove the Prime Geodesic Theorem using standard techniques of analytic number theory.

Proposition 3.6 $\psi(x) = 2x + O(x^{3/4})$

Proof: Let D be the degree of the polynomial $G(s)$, as in the previous section, and suppose $k \geq 2D$ is an integer and $x, c > 1$. Then, by (41) and [18], Theorem 40,

$$\begin{aligned} & \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{R'_{\Gamma, 1}(s)}{R_{\Gamma, 1}(s)} s^{-1} (s+1)^{-1} \cdots (s+k)^{-1} x^s ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\sum_{\gamma} \chi_1(\Gamma_\gamma) l_{\gamma_0} e^{-sl_\gamma} \right) s^{-1} (s+1)^{-1} \cdots (s+k)^{-1} x^s ds \\ &= \frac{1}{k!} \sum_{N(\gamma) \leq x} \chi_1(\Gamma_\gamma) l_{\gamma_0} \left(1 - \frac{N(\gamma)}{x} \right)^k. \end{aligned} \quad (42)$$

Theorem 2.11 tells us that all poles of $R'_\Gamma(s)/R_\Gamma(s)$ lie in the strip $-1 \leq \text{Re}(s) \leq 1$. By virtue of Lemmas 3.3 and 3.5 it is permissible to shift the line of integration in (42) into the half plane $\text{Re}(s) < -1$, taking into account the residues of the poles of $R'_\Gamma(s)/R_\Gamma(s)$. Hence, for $c' < -1$

$$\begin{aligned} & \frac{1}{k!} \sum_{N(\gamma) \leq x} \chi_1(\Gamma_\gamma) l_{\gamma_0} \left(1 - \frac{N(\gamma)}{x} \right)^k \\ &= \sum_{\substack{\alpha \in S_k \\ \text{Re}(\alpha) > c'}} c_k(\alpha) x^\alpha + \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{R'_{\Gamma, 1}(s)}{R_{\Gamma, 1}(s)} s^{-1} (s+1)^{-1} \cdots (s+k)^{-1} x^s ds, \end{aligned} \quad (43)$$

where S_k denotes the set of poles of $(R'_{\Gamma, 1}(s)/R_{\Gamma, 1}(s))s^{-1}(s+1)^{-1} \cdots (s+k)^{-1}$ and $c_k(\alpha)$ denotes the residue at α .

For $x > 1$, Lemma 3.3 implies the integral in (43) tends to zero as $c' \rightarrow -\infty$. If we define

$$\psi_0(x) = \psi(x), \quad \psi_j(x) = \int_0^x \psi_{j-1}(t) dt, \quad j \in \mathbb{N},$$

it is well known that

$$\psi_j(x) = \frac{1}{j!} \sum_{N(\gamma) \leq x} \chi_1(\Gamma_\gamma) l_{\gamma_0} (x - N(\gamma))^j$$

and we deduce from (43) that

$$\psi_k(x) = \sum_{\alpha \in S_k} c_k(\alpha) x^{k+\alpha}. \quad (44)$$

Let $d > 0$. For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ define the operator Δ by setting

$$\Delta f(x) = \sum_{i=0}^{2D} (-1)^i \binom{2D}{i} f(x + (2D - i)d).$$

It follows from (44) and Theorem 2.11, setting $k = 2D$, that

$$\psi_{2D}(x) = \frac{2}{(2D+1)!} x^{2D+1} + \sum_{\alpha \in S_{2D}^{\mathbb{R}}} c_{2D}(\alpha) x^{2D+\alpha} + \sum_{p=-2}^2 \sum_{\alpha \in S_{2D}^{p/4}} c_{2D}(\alpha) x^{2D+\alpha}, \quad (45)$$

where $S_{2D}^{\mathbb{R}} = S_{2D} \cap (\mathbb{R} \setminus \{1\})$, the real elements of S_{2D} not including $\alpha = 1$, and $S_{2D}^q = S_{2D} \cap (q + i(\mathbb{R} \setminus \{0\}))$, the non-real elements of S_{2D} on the line $\text{Re}(s) = q$. The coefficient $((2D+1)!)^{-1}$ on the leading term comes from the fact that $(R'_{\Gamma,1}(s)/R_{\Gamma,1}(s))$ has a double pole at $s = 1$ and from the factors $s^{-1} \dots (s+2D)^{-1}$.

In general if f is at least $2D$ times differentiable,

$$\Delta f(x) = \int_x^{x+d} \int_{t_{2D}}^{t_{2D}+d} \dots \int_{t_2}^{t_2+d} f^{(2D)}(t_1) dt_1 \dots dt_{2D}. \quad (46)$$

By applying the Mean Value Theorem we get

$$\Delta x^r = d^{2D} r(r-1) \dots (r - (2D-1)) \tilde{x}^{r-2D}, \quad (47)$$

where $\tilde{x} \in [x, x + 2Dd]$. In particular we notice that $\Delta(x^{2D+1}) = O(x)$, hence

$$\Delta \left(\frac{2}{(2D+1)!} x^{2D+1} \right) = ax + b$$

for some $a, b \in \mathbb{R}$. Computations show that

$$a = 2 \sum_{j=0}^{2D} (-1)^j \frac{1}{j!(2D-j)!} ((2D-j)d)^{2D} = 2d^{2D}. \quad (48)$$

By definition $\psi_0(x) = \psi_{2D}^{(2D)}(s)$ so from (46) we have

$$\Delta \psi_{2D}(x) = \int_x^{x+d} \int_{t_{2D}}^{t_{2D}+d} \dots \int_{t_2}^{t_2+d} \psi_0(t_1) dt_1 \dots dt_{2D}. \quad (49)$$

By Proposition 1.10, the Euler characteristics $\chi_1(\Gamma_\gamma)$ are all positive. Hence $\psi_0(x)$ is non-decreasing and it follows from (49) that

$$\psi_0(x) \leq d^{-2D} \Delta \psi_{2D}(x) \leq \psi_0(x + 2Dd). \quad (50)$$

It also follows from (47) and (48) that

$$d^{-2D} \Delta \left(\frac{2}{(2D+1)!} x^{2D+1} + \sum_{\alpha \in S_{2D}^{\mathbb{R}}} c_{2D}(\alpha) x^{2D+\alpha} \right) = 2x + O(x^{3/4}).$$

Thus it remains to show that for $p \in \{-2, -1, 0, 1, 2\}$

$$d^{-2D} \Delta \left(\sum_{\alpha \in S_{2D}^{p/4}} c_{2D}(\alpha) x^{2D+\alpha} \right) = O(x^{3/4})$$

and the proposition follows by (50).

Let $p \in \{-2, -1, 0, 1, 2\}$. In order to estimate

$$d^{-2D} \Delta \left(\sum_{\alpha \in S_{2D}^{p/4}} c_{2D}(\alpha) x^{2D+\alpha} \right)$$

we need two estimates for $\Delta(c_{2D}(\alpha)x^{2D+\alpha})$, where $\alpha \in S_{2D}^{p/4}$. The residues at the poles of $R'_{\Gamma,1}(s)/R_{\Gamma,1}(s)$ are $O(1)$, so for $\alpha \in S_{2D}^{p/4}$

$$d^{-2D} \Delta(c_{2D}(\alpha)x^{2D+\alpha}) = O\left(d^{-2D} |\alpha|^{-(2D+1)} x^{2D+p/4}\right).$$

On the other hand, it follows from (47) that

$$d^{-2D} \Delta(c_{2D}(\alpha)x^{2D+\alpha}) = O\left(|\alpha|^{-1} x^{p/4}\right).$$

Define $N_p(t)$ to be the number of poles of $R'_{\Gamma,1}(s)/R_{\Gamma,1}(s)$ on the interval $\frac{p}{4} + i(0, t]$. From Lemma 3.4 we have that $N_p(t) = O(t^D)$. Thus

$$\begin{aligned} & d^{-2D} \Delta \left(\sum_{\alpha \in S_{2D}^{p/4}} c_{2D}(\alpha) x^{2D+\alpha} \right) \\ &= O \left(x^{p/4} \int_1^{K^D} t^{-1} dN(t) + d^{-2D} x^{2D+p/4} \int_{K^D}^{\infty} t^{-(2D+1)} dN(t) \right) \\ &= O \left(K^{D-1} x^{p/4} + K^{-(D+2)} d^{-2D} x^{2+p/4} \right) \end{aligned} \tag{51}$$

If we choose K and d appropriately, then we can conclude from (51) that $d^{-2} \Delta \left(\sum_{\alpha \in S_2^{p/4}} c_2(\alpha) x^{2+\alpha} \right) = O(x^{3/4})$, as required. \square

Proposition 3.7 $\tilde{\psi}(x) = 8x + O(x^{3/4})$

Proof: The proof follows exactly as for the previous proposition, replacing $R_{\Gamma,1}(s)$ with $R_{\Gamma,\tilde{\sigma}}(s)$ throughout. It follows in particular from the fact (Theorem 2.11) that $R_{\Gamma,\tilde{\sigma}}(s)$ has a zero of order eight at the point $s = 1$ and from the fact (Lemma 2.10) that $\text{tr } \tilde{\sigma}(b_\gamma) \geq 0$ for all $\gamma \in \mathcal{E}_P^p(\Gamma)$ and $\text{tr } \tilde{\sigma}(b_\gamma) = 0$ if and only if γ is non-regular. \square

3.3 The Wiener-Ikehara Theorem

We shall use the following version of the Wiener-Ikehara theorem (see also [5], Chapter XI, Theorem 2, and [10], Theorem 3.2).

Let $R_k(s)$, $k \in \mathbb{N}$ be a sequence of rational functions on \mathbb{C} . for an open set $U \subset \mathbb{C}$ let $\mathbb{N}(U)$ be the set of natural numbers k such that the pole divisor of R_k does not intersect U . We say that the series

$$\sum_{k \in \mathbb{N}} R_k(s)$$

converges weakly locally uniformly on \mathbb{C} if for every open $U \subset \mathbb{C}$ the series

$$\sum_{k \in \mathbb{N}(U)} R_k(s)$$

converges locally uniformly on U .

Theorem 3.8 (Wiener-Ikehara) *Let $A(x) \geq 0$ be a monotonic measurable function on \mathbb{R}_+ . Suppose that the integral*

$$L(s) = \int_0^\infty A(x) e^{-sx} dx$$

converges for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$. Suppose further that there are $j \in \mathbb{N}$, $r \in \mathbb{R}$ and a countable set I , and for each $i \in I$ there is $\theta_i \in \mathbb{C}$ with $\operatorname{Re}(\theta_i) < 1$ and $c_i \in \mathbb{Z}$, such that the function

$$L(s) - \left(\frac{\partial}{\partial s}\right)^{j+1} \frac{r}{s-1} - \sum_{i \in I} c_i \left(\frac{\partial}{\partial s}\right)^{j+1} \frac{1}{s-\theta_i}$$

extends to a holomorphic function on the half-plane $\operatorname{Re}(s) \geq 1$. Here we assume the sum converges weakly locally uniformly absolutely on \mathbb{C} . Then

$$\lim_{x \rightarrow \infty} A(x)x^{-(j+1)}e^{-x} = r.$$

Proof: This is a straightforward variation of the proof of the Wiener-Ikehara-Theorem in [5]. \square

3.4 The Dirichlet series

Let B^0 be a closed subset of the compact group B with the following properties: it is of measure zero; it is invariant under the map $b \mapsto b^{-1}$ and contains all fixed points of this map; and its complement $B^1 = B \setminus B^0$ in B is homeomorphic to an open subset of Euclidean space each of whose connected components is contractible.

The Weyl group $W = W(M, B)$ contains two elements and the non-trivial element acts on B by $b \mapsto b^{-1}$, so the invariance condition above says that both B^0 and B^1 are invariant under the action of W . The fixed points under the action of the Weyl group are precisely the non-regular elements of B , so the assumption that B^0 contains all these fixed points is equivalent to $B^{\text{reg}} \subset B^0$, where B^{reg} denotes the subset of nonregular elements of B . The action of W on B^1 permutes the connected components and the assumption $B^{\text{reg}} \subset B^0$ implies that the quotient space B^1/W is also homeomorphic to an open subset of Euclidean space each of whose connected components is simply connected.

For subsets S and T of G we denote by $S.T$ the subset

$$S.T = \{sts^{-1} : s \in S, t \in T\}$$

of G . Let $M_{\text{ell}}^1 = M.B^1 \subset M$. Let $\mathcal{E}_P^0(\Gamma)$ and $\mathcal{E}_P^1(\Gamma)$ be the subsets of $\mathcal{E}_P(\Gamma)$ consisting of all conjugacy classes $[\gamma]$ such that $b_\gamma \in B^0$ or $b_\gamma \in B^1$ respectively. The assumption that B^0 is of measure zero is not required for the following lemma, but will be needed later on.

Lemma 3.9 *There exist a set $M_c \subset M_{\text{ell}}^1$ with compact closure in M and a monotonically increasing sequence (g_n) of smooth functions on M , supported on M_c , which are invariant under conjugation by elements of K_M , and whose orbital integrals satisfy*

$$\mathcal{O}_{b_\gamma}^M(g_n) = \int_{M/B} g_n(xb_\gamma x^{-1}) dx \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

for all $\gamma \in \mathcal{E}_P^1(\Gamma)$.

Proof: We view M_{ell}^1 as a fibre bundle with base space B^1/W and fibres homeomorphic to M/B .

Let $d(\cdot, \cdot)$ denote the metric on B given by the form b in (12). For $n \in \mathbb{N}$ let $\tilde{B}_n \subset B^1$ be the set $\tilde{B}_n = \{b \in B : d(b, B^0) \geq 1/n\}$ and let $B_n = \tilde{B}_n/W$. Then, by [41], Corollary 1.11, for each $n \in \mathbb{N}$ there exists a function $h_n : B/W \rightarrow \mathbb{R}$ such that $0 \leq h_n(b) \leq 1$ for all $b \in B/W$, for all $b \in B_n$ we have $h_n(b) = 1$ and h_n is supported on B^1/W . We may assume that the h_n 's form an increasing series.

Let U be a compact neighbourhood of a point in M/B homeomorphic to a subset of Euclidean space. Let $k : M/B \rightarrow \mathbb{R}$ be a smooth positive function supported on U and satisfying $\int_{M/B} k(m) dm = 1$.

We now define, for $n \in \mathbb{N}$, functions $\tilde{g}_n : M \rightarrow \mathbb{R}$ as follows. On each connected component V of B^1/W we fix a trivialisation of the restriction to V of the bundle $M/B \rightarrow M_{\text{ell}}^1 \rightarrow B^1/W$. Then for $v \in V$ and $m \in M/B$ define $\tilde{g}_n(mvm^{-1}) = h_n(v)k(m)$. Since V is simply connected there are no problems with global agreement of this definition. For $m \in M \setminus M_{\text{ell}}^1$ define $\tilde{g}_n(m) = 0$. Finally we define the functions $g_n : M \rightarrow \mathbb{R}$ by

$$g_n(m) = \frac{1}{2} \int_{K_M} \tilde{g}_n(kmk^{-1}) dk$$

for all $m \in M$. Then the functions g_n form an increasing sequence of smooth K_M -invariant functions supported on the compact set M_c , which we define to be the closure in M of $K_M \cdot (U \cdot B^1)$. We will show that their orbital integrals have the required properties. Let $b \in B^1$.

$$\begin{aligned}
\mathcal{O}_b^M(g_n) &= \int_{M/B} g_n(mbm^{-1}) dm \\
&= \frac{1}{2} \int_{M/B} \int_{K_M} \tilde{g}_n(kmbm^{-1}k^{-1}) dk dm \\
&= \frac{1}{2} \int_{M/B} \int_{K_M} \tilde{g}_n(mkbbk^{-1}m^{-1}) dk dm \\
&= \frac{1}{2} \int_{M/B} \int_B \tilde{g}_n(mbm^{-1}) + \tilde{g}_n(mb^{-1}m^{-1}) dk dm \\
&= \frac{1}{2} (h_n(b) + h_n(b^{-1})) \int_{M/B} k(m) dm \\
&= h_n(b).
\end{aligned}$$

The last equality holds since $bW = b^{-1}W$ in B^1/W . There exists $N \in \mathbb{N}$ such that $b \in B_n$ for all $n \geq N$, hence $\mathcal{O}_b^M(g_n) \rightarrow 1$ as $n \rightarrow \infty$, by the definition of the functions h_n . \square

For $n \in \mathbb{N}$ let

$$L_n^j(s) = \sum_{[\gamma] \in \mathcal{E}_P^1(\Gamma)} \chi_1(\Gamma_{\gamma_0}) \mathcal{O}_{b_\gamma}(g_n) l_{\gamma_0} \frac{l_\gamma^{j+1} e^{-sl_\gamma}}{\det(1 - (a_\gamma b_\gamma)^{-1} | \mathfrak{n})},$$

let

$$r_n = \int_M g_n(x) dx,$$

and let

$$m_{n,\lambda} = \sum_{q=0}^4 (-1)^q r_n \dim H^q(\mathfrak{n}, \pi_K)^\lambda.$$

Proposition 3.10 *For all $n \in \mathbb{N}$ and for $j \in \mathbb{N}$ large enough the series $L_n^j(s)$ converges locally uniformly in the set $\{s \in \mathbb{C} : \operatorname{Re}(s) > 1\}$.*

The function $L_n^j(s)$ can be written as a Mittag-Leffler series

$$L_n^j(s) = \left(\frac{\partial}{\partial s} \right)^{j+1} \frac{r_n}{s-1} + \sum_{\pi \in \hat{G}} N_\Gamma(\pi) \sum_{\lambda \in \mathfrak{a}^*} m_{n,\lambda} \left(\frac{\partial}{\partial s} \right)^{j+1} \frac{1}{s - \lambda(H_1)},$$

where the summand of the double series corresponding to $\pi = \operatorname{triv}$, $\lambda = -2\rho_P$ is excluded. The double series converges weakly locally uniformly on \mathbb{C} . For $\pi \in \hat{G}$, $\lambda \in \mathfrak{a}^*$ such that $(\pi, \lambda) \neq (\operatorname{triv}, -2\rho_P)$ we have $m_{n,\lambda} \neq 0$ only if $\operatorname{Re}(\lambda) > -2\rho_P$. Thus, in particular, the double series converges locally uniformly on $\{s \in \mathbb{C} : \operatorname{Re}(s) > 1\}$.

Proof: We can put $f = g_n$ in (14) to define the function $\Phi_{g_n, j, s}$, which by Proposition 2.3, for j and $\operatorname{Re}(s)$ large enough, goes into the Selberg trace formula to give the equation

$$\sum_{\pi \in \hat{G}} N_\Gamma(\pi) \operatorname{tr} \pi(\Phi_{g_n}) = \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_{b_\gamma}(g_n) \frac{l_\gamma^{j+1} e^{-sl_\gamma}}{\det(1 - (a_\gamma b_\gamma)^{-1} | \mathfrak{n})}.$$

By the definition of the functions g_n , the orbital integrals $\mathcal{O}_{b_\gamma}(g_n)$ are equal to zero when $\gamma \in \mathcal{E}_P^0(\Gamma)$. Also, for $\gamma \in \mathcal{E}_P^1(\Gamma)$, since we have assumed that $B^{\operatorname{reg}} \subset B^0$ we have that γ is regular and we get from Lemma 1.9 that

$$\operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) = \chi_1(\Gamma_\gamma) l_{\gamma_0}.$$

Thus we can see that the geometric side of the trace formula is equal to $L_n^j(s)$. It follows that for j and $\operatorname{Re}(s)$ large enough the Dirichlet series $L_n^j(s)$ converges absolutely.

Lemma 3.11

$$\mathrm{tr} \pi(\Phi_{g_n}) = (-1)^{j+1} \left(\frac{\partial}{\partial s} \right)^{j+1} r_n \sum_{\lambda \in \mathfrak{a}^*} \dim H^\bullet(\mathfrak{n}, \pi_K)^\lambda \frac{1}{s - \lambda(H_1)}.$$

Proof: Replacing f_σ with g_n in (22) we get

$$\begin{aligned} \mathrm{tr} \pi(\Phi_{g_n}) &= \int_{MA^-} g_n(m) \Theta_{H^\bullet(\mathfrak{n}, \pi_K)}^{MA}(ma) dm g_s^j(a) da \\ &= \int_{A^-} \int_M g_n(m) dm \mathrm{tr} (a | H^\bullet(\mathfrak{n}, \pi_K)) l_a^{j+1} e^{-sl_a} da. \\ &= r_n \int_0^\infty \sum_{\lambda \in \mathfrak{a}^*} \dim H^\bullet(\mathfrak{n}, \pi_K)^\lambda e^{(\lambda(H_1) - s)t} t^{j+1} dt \\ &= (-1)^{j+1} \left(\frac{\partial}{\partial s} \right)^{j+1} r_n \sum_{\lambda \in \mathfrak{a}^*} \dim H^\bullet(\mathfrak{n}, \pi_K)^\lambda \frac{1}{s - \lambda(H_1)}. \end{aligned}$$

□

By the above lemma and Proposition 2.15, and using similar arguments to those used in the proof of Proposition 2.17, we can see that the spectral side of the trace formula is equal to the Mittag-Leffler series given in the proposition.

Since $L_n^j(s)$ is a Dirichlet series with positive coefficients it will converge locally uniformly for s in some open set. By proving the convergence of the Mittag-Leffler series we shall show that the Dirichlet series extends to a holomorphic function on $\{\mathrm{Re}(s) > 1\}$ and hence, since it has positive coefficients, converges locally uniformly there.

We recall that for $\lambda \in \mathfrak{a}^*$ we write $\|\lambda\|$ for the norm given by the form b in (12). It then follows from Proposition 2.5 that there exist $m_1 \in \mathbb{N}$ and $C > 0$ such that for every $\pi \in \hat{G}$ and every $\lambda \in \mathfrak{a}^*$ we have

$$|m_{n,\lambda}| \leq C(1 + \|\lambda\|)^{m_1}.$$

If S denotes the set of all pairs $(\pi, \lambda) \in \hat{G} \times \mathfrak{a}^*$ such that $m_{n,\lambda} \neq 0$, then there exists $m_2 \in \mathbb{N}$ such that

$$\sum_{(\pi, \lambda) \in S} \frac{N_\Gamma(\pi)}{(1 + \|\lambda\|)^{m_2}} < \infty.$$

Now, let $U \subset \mathbb{C}$ be open and let $S(U)$ be the set of all pairs $(\pi, \lambda) \in \hat{G} \times \mathfrak{a}^*$ such that $m_{n,\lambda} \neq 0$ and $\lambda(H_1) \notin U$. Let V be a compact subset of U . We have to show that for some $j \in \mathbb{N}$, which does not depend on U or V ,

$$\sup_{s \in V} \sum_{(\pi, \lambda) \in S(U)} \left| \frac{N_\Gamma(\pi) m_{n,\lambda}}{(s - \lambda(H_1))^{j+2}} \right| < \infty.$$

Let m_1, m_2 be as above and let $j \geq m_1 + m_2 - 2$. Since $V \subset U$ and V is compact there is $\varepsilon > 0$ such that $s \in V$ and $(\pi, \lambda) \in S(U)$ implies $|s - \lambda(H_1)| \geq \varepsilon$. Hence there is $c > 0$ such that for every $s \in V$ and every $(\pi, \lambda) \in S(U)$,

$$|s - \lambda(H_1)| \geq c(1 + \|\lambda\|).$$

Putting this all together we get

$$\begin{aligned} \sup_{s \in V} \sum_{(\pi, \lambda) \in S(U)} \left| \frac{N_\Gamma(\pi) m_{n,\lambda}}{(s - \lambda(H_1))^{j+2}} \right| &\leq \sup_{s \in V} \sum_{(\pi, \lambda) \in S(U)} \frac{1}{c^{j+2}} \frac{N_\Gamma(\pi) |m_{n,\lambda}|}{(1 + \|\lambda\|)^{j+2}} \\ &\leq \sup_{s \in V} \sum_{(\pi, \lambda) \in S(U)} \frac{C}{c^{j+2}} \frac{N_\Gamma(\pi)}{(1 + \|\lambda\|)^{m_2}} \\ &< \infty, \end{aligned}$$

which proves the proposition. □

3.5 Estimating $\psi_n(x)$

Let

$$\phi_n^j(x) = \sum_{\substack{[\gamma] \in \mathcal{E}_P^1(\Gamma) \\ N(\gamma) \leq x}} \chi_1(\Gamma_{\gamma_0}) \mathcal{O}_{b_\gamma}(g_n) l_{\gamma_0} \frac{l_\gamma^{j+1}}{\det(1 - (a_\gamma b_\gamma)^{-1} | \mathbf{n})}.$$

Lemma 3.12

$$\int_0^\infty \phi_n^j(e^x) e^{-sx} dx = L_n^j(s)$$

Proof: This follows from Abel's summation formula ([5], Chapter VII, Theorem 6). \square

Let

$$\phi_n(x) = \sum_{\substack{[\gamma] \in \mathcal{E}_P^1(\Gamma) \\ N(\gamma) \leq x}} \chi_1(\Gamma_{\gamma_0}) \mathcal{O}_{b_\gamma}(g_n) l_{\gamma_0} \frac{1}{\det(1 - (a_\gamma b_\gamma)^{-1} | \mathbf{n})}.$$

Lemma 3.13 For each $n \in \mathbb{N}$ we have

$$\phi_n(x) \sim r_n x.$$

Proof: By Propostion 3.10 and Lemma 3.12 we can apply Theorem 3.8 to the series $L_n^j(s)$ and the function $\phi_n^j(x)$ to deduce that

$$\lim_{x \rightarrow \infty} \frac{\phi_n^j(x)}{x(\log x)^{j+1}} = r_n. \quad (52)$$

Also it is clear that

$$\phi_n(x) \geq \frac{\phi_n^j(x)}{(\log x)^{j+1}},$$

so it follows that

$$\liminf_{x \rightarrow \infty} \frac{\phi_n(x)}{x} \geq r_n.$$

Let $0 < \mu < 1$. Then

$$\begin{aligned} \phi_n^j(x) &\geq \sum_{\substack{[\gamma] \in \mathcal{E}_P^1(\Gamma) \\ x^\mu < N(\gamma) \leq x}} \chi_1(\Gamma_{\gamma_0}) \mathcal{O}_{b_\gamma}(g_n) l_{\gamma_0} \frac{l_\gamma^{j+1}}{\det(1 - (a_\gamma b_\gamma)^{-1} | \mathbf{n})} \\ &\geq \mu^{j+1} (\log x)^{j+1} \sum_{\substack{[\gamma] \in \mathcal{E}_P^1(\Gamma) \\ x^\mu < N(\gamma) \leq x}} \chi_1(\Gamma_{\gamma_0}) \mathcal{O}_{b_\gamma}(g_n) l_{\gamma_0} \frac{1}{\det(1 - (a_\gamma b_\gamma)^{-1} | \mathbf{n})} \\ &= \mu^{j+1} (\log x)^{j+1} (\phi_n(x) - \phi_n(x^\mu)). \end{aligned}$$

From this we get

$$\frac{\phi_n(x)}{x} \leq \mu^{-(j+1)} \frac{\phi_n^j(x)}{x(\log x)^{j+1}} + \frac{\phi_n(x^\mu)}{x^\mu x^{1-\mu}}. \quad (53)$$

Assume first that $\phi_n(x)/x$ tends to infinity as $x \rightarrow \infty$. Then there is a sequence x_m of positive real numbers, tending to infinity such that $\phi(x_m)/x_m$ tends to infinity as $m \rightarrow \infty$ and

$$\frac{\phi_n(x_m)}{x_m} \geq \frac{\phi_n(x_m^\mu)}{x_m^\mu}$$

for all m . Then

$$\frac{\phi_n(x_m)}{x_m} \leq \mu^{-(j+1)} \frac{\phi_n^j(x_m)}{x_m (\log x_m)^{j+1}} + \frac{\phi_n(x_m)}{x_m} \cdot \frac{1}{x_m^{1-\mu}},$$

so that

$$\frac{\phi_n(x_m)}{x_m} \leq \frac{\mu^{-(j+1)} \frac{\phi_n^j(x_m)}{x_m (\log x_m)^{j+1}}}{1 - \frac{1}{(x_m)^{1-\mu}}}.$$

By (52) the right hand side converges, so we have a contradiction. This implies that $\lim_{x \rightarrow \infty} \frac{\phi_n(x)}{x}$ exists and is finite and we set

$$L = \limsup_{x \rightarrow \infty} \frac{\phi_n(x)}{x} = \limsup_{x \rightarrow \infty} \frac{\phi_n(x^\mu)}{x^\mu}.$$

From (52) and (53) we get

$$L \leq r_n \mu^{-(j+1)} + L \limsup_{x \rightarrow \infty} \frac{1}{x^{1-\mu}} = r_n \mu^{-(j+1)}.$$

Since this holds for any value of μ in the interval $0 < \mu < 1$ we get that $L \leq r$ and the lemma follows. \square

For $n \in \mathbb{N}$ and $x > 0$ let

$$\psi_n(x) = \sum_{\substack{[\gamma] \in \mathcal{E}_P^1(\Gamma) \\ N(\gamma) \leq x}} \chi_1(\Gamma_{\gamma_0}) \mathcal{O}_{b_\gamma}(g_n) l_{\gamma_0}.$$

Proposition 3.14 *For each $n \in \mathbb{N}$ we have $\psi_n(x) \sim r_n x$.*

Proof: We note that $0 < \det(1 - a_\gamma b_\gamma | \mathbf{n}) < 1$ for all $\gamma \in \mathcal{E}_P^{\text{reg}}(\Gamma)$ and that the value of the determinant tends to 1 as l_γ tends to infinity, hence for $0 < \varepsilon < 1$ there are only finitely many $\gamma \in \mathcal{E}_P^{\text{reg}}(\Gamma)$ such that $\det(1 - a_\gamma b_\gamma | \mathbf{n}) \leq 1 - \varepsilon$. Since $\mathcal{E}_P^1(\Gamma) \subset \mathcal{E}_P^{\text{reg}}(\Gamma)$ the same holds for $\gamma \in \mathcal{E}_P^1(\Gamma)$. We fix $0 < \varepsilon < 1$ and define for each $n \in \mathbb{N}$ the functions $\phi_{n,\varepsilon}$ and $\psi_{n,\varepsilon}$ to be the same sums as for ϕ_n and ψ_n respectively but restricted to those classes $[\gamma] \in \mathcal{E}_P^1(\Gamma)$ such that $1 - \varepsilon < \det(1 - a_\gamma b_\gamma | \mathbf{n}) < 1$. It then follows that both

$$\frac{\phi_n(x) - \phi_{n,\varepsilon}(x)}{x} \rightarrow 0 \tag{54}$$

and

$$\frac{\psi_n(x) - \psi_{n,\varepsilon}(x)}{x} \rightarrow 0 \tag{55}$$

as $x \rightarrow \infty$. Now

$$1 - \varepsilon < \det(1 - a_\gamma b_\gamma | \mathbf{n}) < 1$$

immediately implies

$$\frac{1 - \varepsilon}{\det(1 - a_\gamma b_\gamma | \mathbf{n})} < 1 < \frac{1}{\det(1 - a_\gamma b_\gamma | \mathbf{n})},$$

so summing up we get

$$\frac{\phi_{n,\varepsilon}(x)}{x} (1 - \varepsilon) < \frac{\psi_{n,\varepsilon}(x)}{x} < \frac{\phi_{n,\varepsilon}(x)}{x}.$$

By Lemma 3.13 and (54), it then follows that

$$r_n(1 - \varepsilon) \leq \liminf_{x \rightarrow \infty} \frac{\psi_{n,\varepsilon}(x)}{x} \leq \limsup_{x \rightarrow \infty} \frac{\psi_{n,\varepsilon}(x)}{x} \leq r_n.$$

It then follows from (55) that

$$r_n(1 - \varepsilon) \leq \liminf_{x \rightarrow \infty} \frac{\psi_n(x)}{x} \leq \limsup_{x \rightarrow \infty} \frac{\psi_n(x)}{x} \leq r_n.$$

Since this holds for any value of ε in the interval $0 < \varepsilon < 1$ the proposition is proven. \square

3.6 Estimating $\psi^1(x)$

Lemma 3.15 *The sequence (r_n) is monotonically increasing and $r_n \rightarrow 2$ as $n \rightarrow \infty$.*

Proof: That the sequence (r_n) is monotonically increasing is clear since the sequence of functions (g_n) is monotonically increasing.

By Weyl's integral formula ([26], Proposition 5.27), and since the functions g_n are only non-zero on elliptic elements of M , we have

$$\begin{aligned} r_n &= \int_M g_n(x) dx \\ &= \frac{1}{|W(B : G)|} \int_B \int_{M/B} g_n(xbx^{-1}) |D(b)|^2 dx db \\ &= \frac{1}{|W(B : G)|} \int_B \mathcal{O}_b^M(g_n) |D(b)|^2 db, \end{aligned}$$

where $W(B : G)$ is the Weyl group and $D(b)$ is the Weyl denominator. Since the sequence of functions g_n is monotonically increasing and the functions are all supported within a given compact subset of M we can interchange integral and limit to get

$$\begin{aligned} \lim_{n \rightarrow \infty} r_n &= \lim_{n \rightarrow \infty} \frac{1}{|W(B : G)|} \int_B \mathcal{O}_b^M(g_n) |D(b)|^2 db \\ &= \frac{1}{|W(B : G)|} \int_B \lim_{n \rightarrow \infty} \mathcal{O}_b^M(g_n) |D(b)|^2 db. \end{aligned}$$

Furthermore, the orbital integrals $\mathcal{O}_b^M(g_n)$ tend to one as $n \rightarrow \infty$, except for b in a set of measure zero, so it follows that

$$\lim_{n \rightarrow \infty} r_n = \frac{1}{|W(B : G)|} \int_B |D(b)|^2 db.$$

The Weyl group $W(B : G)$ consists of the identity element and the element $\text{diag}(1, -1, 1, -1)$. We can also compute the Weyl denominator $D(b)$ for $b \in B$. Let

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in \text{SO}(2)$$

and let

$$R(\theta, \phi) = \begin{pmatrix} R(\theta) & \\ & R(\phi) \end{pmatrix} \in B.$$

Then

$$\begin{aligned} D(R(\theta, \phi)) &= e^{i(\theta+\phi)}(1 - e^{-2i\theta})(1 - e^{-2i\phi}) \\ &= (e^{i(\theta+\phi)} + e^{-i(\theta+\phi)}) - (e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}) \\ &= 2\cos(\theta + \phi) - 2\cos(\theta - \phi) \\ &= 2(\cos \theta \cos \phi - \sin \theta \sin \phi - \cos \theta \cos \phi - \sin \theta \sin \phi) \\ &= -4 \sin \theta \sin \phi \end{aligned}$$

We have normalised the Haar measure on B so that $\int_B db = 1$, hence

$$\begin{aligned} \frac{1}{|W(B : G)|} \int_B |D(b)|^2 db &= \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} 16 \sin^2 \theta \sin^2 \phi \frac{d\theta d\phi}{4\pi^2} \\ &= \frac{2}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sin^2 \theta \sin^2 \phi d\theta d\phi \\ &= \frac{2}{\pi^2} \cdot \pi^2 = 2. \end{aligned}$$

This proves the lemma. □

Let

$$\psi^1(x) = \sum_{\substack{[\gamma] \in \mathcal{S}_P^1(\Gamma) \\ N(\gamma) \leq x}} \chi_1(\Gamma_{\gamma_0}) l_{\gamma_0}.$$

Proposition 3.16 $\psi^1(x) \sim \frac{2x}{\log x}.$

Proof: For all $n \in \mathbb{N}$ and all $x > 0$ we have $\psi_n(x) \leq \psi^1(x) \leq \psi(x)$. It then follows, using Proposition 3.6 and Proposition 3.14, that for all $n \in \mathbb{N}$

$$r_n = \liminf_{x \rightarrow \infty} \frac{\psi_n(x)}{x} \leq \liminf_{x \rightarrow \infty} \frac{\psi^1(x)}{x} \leq \limsup_{x \rightarrow \infty} \frac{\psi^1(x)}{x} \leq \limsup_{x \rightarrow \infty} \frac{\psi(x)}{x} = 2.$$

We can then deduce from Lemma 3.15 that $\lim_{x \rightarrow \infty} \frac{\psi^1(x)}{x} = 2$. \square

3.7 Estimating $\pi(x)$, $\tilde{\pi}(x)$ and $\pi^1(x)$

To keep the notation less cluttered, in the sums over conjugacy classes in Γ which appear in this section we shall not specify which set of classes the sum is being taken over, since this will be clear from the context. We shall always use γ_0 to denote primitive elements and where γ and γ_0 appear together in the same formula we shall mean that γ_0 is the primitive element underlying γ .

Proposition 3.17

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{2x/\log x} = \lim_{x \rightarrow \infty} \frac{\psi(x)}{2x} = 1.$$

Proof: We can write

$$\psi(x) = \sum_{N(\gamma_0) \leq x} n_{\gamma_0} \chi_1(\Gamma_{\gamma_0}) l_{\gamma_0},$$

where $n_{\gamma_0} \in \mathbb{N}$ is maximal such that $N(\gamma_0)^{n_{\gamma_0}} \leq x$. By definition $N(\gamma_0) = e^{l_{\gamma_0}}$, so $N(\gamma_0)^n \leq x$ implies $nl_{\gamma_0} \leq \log x$ and we can see that

$$\psi(x) \leq \log(x) \pi(x). \quad (56)$$

Next we fix a real number $0 < a < 1$. By Proposition 1.10 the Euler characteristic $\chi_1(\Gamma_{\gamma_0}) > 0$ for all $\gamma_0 \in \mathcal{E}_P^p(\Gamma)$, so for $x > 1$,

$$\psi(x) \geq \sum_{x^a < N(\gamma_0) \leq x} \chi_1(\Gamma_{\gamma_0}) l_{\gamma_0}.$$

As above, $N(\gamma_0) > x^a$ implies $l_{\gamma_0} > \log x^a$, hence

$$\psi(x) \geq a \log x \sum_{x^a < N(\gamma_0) \leq x} \chi_1(\Gamma_{\gamma_0}) = a \log x (\pi(x) - \pi(x^a)).$$

Since $\Gamma \subset G$ is discrete, $\pi(x^a) < Cx^a$ for some constant C so

$$\psi(x) > a\pi(x) \log x - aCx^a \log x,$$

which gives

$$\frac{\psi(x)}{2x} > a\pi(x) \frac{\log x}{2x} - aC \frac{\log x}{2x^{1-a}}.$$

Since $0 < a < 1$, it follows that $(\log x)/x^{1-a} \rightarrow 0$ as $x \rightarrow \infty$ and

$$\lim_{x \rightarrow \infty} \frac{\psi(x)}{2x} \geq a \lim_{x \rightarrow \infty} \frac{\pi(x)}{2x/\log x}$$

for all $0 < a < 1$. Hence

$$\lim_{x \rightarrow \infty} \frac{\psi(x)}{2x} \geq \lim_{x \rightarrow \infty} \frac{\pi(x)}{2x/\log x}.$$

Together with (56) and Proposition 3.6 this proves the proposition. \square

Proposition 3.18

$$\lim_{x \rightarrow \infty} \frac{\tilde{\pi}(x)}{8x/\log x} = \lim_{x \rightarrow \infty} \frac{\tilde{\psi}(x)}{8x} = 1.$$

Proof: Exactly as for the previous proposition, making use of Proposition 3.7 and Lemma 2.10. \square

Proposition 3.19

$$\pi(x) = 2\text{li}(x) + O\left(\frac{x^{3/4}}{\log x}\right).$$

Proof: We consider the function

$$S(x) = \sum_{N(\gamma) \leq x} \chi_1(\Gamma_{\gamma_0}) \frac{l_{\gamma_0}}{l_{\gamma}} = \sum_{N(\gamma_0) \leq x} \chi_1(\Gamma_{\gamma_0}) + \sum_{k \geq 2} \sum_{N(\gamma_0) \leq x^{1/k}} \chi_1(\Gamma_{\gamma_0}) \frac{1}{k}.$$

We consider the double sum on the right. Since $\Gamma \subset G$ is discrete there is a geodesic γ_{\min} of minimum length. For a given x the inner sum contains at least one summand only for $k \leq \log x / l_{\gamma_{\min}}$. For each such $k \geq 2$, by Proposition 3.17, the inner sum is equal to $O(\sqrt{x} / \log x)$. Therefore we have

$$S(x) = \pi(x) + O(\sqrt{x}). \quad (57)$$

Now

$$\begin{aligned} \int_2^x \frac{\psi(t)}{t \log^2 t} dt &= \int_2^x \sum_{N(\gamma) \leq t} \chi_1(\Gamma_{\gamma_0}) \frac{l_{\gamma_0}}{t \log^2 t} dt \\ &= \sum_{N(\gamma) \leq x} \int_{N(\gamma)}^x \chi_1(\Gamma_{\gamma_0}) \frac{l_{\gamma_0}}{t \log^2 t} dt \\ &= \sum_{N(\gamma) \leq x} \chi_1(\Gamma_{\gamma_0}) l_{\gamma_0} \left(\frac{1}{l_{\gamma}} - \frac{1}{\log x} \right) \\ &= S(x) - \frac{\psi(x)}{\log x}. \end{aligned}$$

Hence $S(x) = \int_2^x \frac{\psi(t)}{t \log^2 t} dt + \frac{\psi(x)}{\log x}$. By Proposition 3.6 we have

$$\begin{aligned} S(x) &= \int_2^x \frac{2}{\log^2 t} dt + \frac{2x}{\log x} + O\left(\int_2^x \frac{1}{t^{1/4} \log^2 t} dt\right) + O\left(\frac{x^{3/4}}{\log x}\right) \\ &= \left[-\frac{2t}{\log t}\right]_2^x + \int_2^x \frac{2}{\log t} dt + \frac{2x}{\log x} + O\left(\frac{x^{3/4}}{\log x}\right) \\ &= \int_2^x \frac{2}{\log t} dt + O\left(\frac{x^{3/4}}{\log x}\right). \end{aligned}$$

Together with (57) this proves the proposition. \square

Proposition 3.20

$$\tilde{\pi}(x) = 8\text{li}(x) + O\left(\frac{x^{3/4}}{\log x}\right).$$

Proof: Exactly as for the previous proposition, making use of Proposition 3.7 and Proposition 3.18. \square

The proof of Theorem 3.2 proceeds exactly as for the proof of Proposition 3.17, making use of Proposition 3.16.

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